10 Differential equations

10.1 Introduction and definitions

A **differential equation (DE)** is a relation between variables (e.g. x, y) and their derivatives (e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}$). For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 3x^4 - t$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 3xy = e^y$ are all differential equations.

Because of the link between derivatives and rates of change, differential equations can be found in many fields: physics (laws of motion), chemistry (chemical kinetics), biology (population growth), economics (modelling of financial instruments) and engineering (fluid dynamics) are just a few examples of where differential equations play a big role in our understanding of the world.

Our aim for this topic is to find solutions to DEs. A **solution** to a DE is a relation between our variables (e.g. x, y) that satisfy the DE. For example, $y = x^2$ is a solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

There are often many solutions to a given DE. In the previous example, $y = x^2 + 1$ is another solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. The **general solution** refers to a solution that includes all the possible solutions of a DE. For $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, the general solution is $y = x^2 + c$, where c is an arbitrary constant.

A **particular solution** refers to one specific solution of a DE. For example, for $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, if we are further given that $y = 3$ when $x = 0$, then $y = x^2 + 3$ is the particular solution.

10.2 Verifying solutions

We can verify if a given equation is a solution of a DE by differentiation.

Example 1. Verify that $y = x^3 e^{x^2}$ is a solution of the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} = y(3 + 2x^2)$.

Solution 1. Differentiating $y = x^3 e^{x^2}$ w.r.t. x (using the product rule),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 e^{x^2} + x^3 \left(2xe^{x^2}\right)$$

Multiplying x on both sides,

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^3 e^{x^2} + 2x^2 \left(x^3 e^{x^2}\right)$$

By observing that $y = x^3 e^{x^2}$, we have

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3y + 2x^2 y \\ &= y(3 + 2x^2) \end{aligned}$$

This verifies that $y = x^3 e^{x^2}$ is a solution. ■

10.3 Solving DEs of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

The simplest DEs are of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x}$ depends only on x. We can obtain the general solution by integrating $f(x)$ (there are infinitely many solutions due to the constant of integration).

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \quad \Longrightarrow \quad y = \int f(x)\,\mathrm{d}x}$$

Example 2. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2+1}$.

(a) Find the general solution.

(b) Find the particular solution, given that $y = 2$ when $x = 0$.

Solution 2.

(a)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2+1}$$
$$y = \int \frac{1}{x^2+1}\,\mathrm{d}x$$
$$y = \tan^{-1} x + c$$

General solution: $y = \tan^{-1} x + c$, where c is an arbitrary constant. ■

(b) When $x = 0$, $y = 2$

$2 = \tan^{-1} 0 + c$

$c = 2$

Particular solution: $y = \tan^{-1} x + 2$. ■

10.4 Solving DEs of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(y)$

DEs of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(y)$ where $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ depends only on y can also be solved by integration. The trick is to move $f(y)$ over to the left to get $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$. We integrate both sides w.r.t. x to obtain $\displaystyle\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 1\,\mathrm{d}x$. This simplifies to $\displaystyle\int \frac{1}{f(y)}\,\mathrm{d}y = \int 1\,\mathrm{d}x$.

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(y) \quad\Longrightarrow\quad \int \frac{1}{f(y)}\,\mathrm{d}y = \int 1\,\mathrm{d}x}$$

Example 3. Find the general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2y$.

Solution 3.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - 2y \\ \int \frac{1}{1-2y}\,\mathrm{d}y &= \int 1\,\mathrm{d}x \\ \frac{\ln|1-2y|}{-2} &= x + c', \text{ where } c' \text{ is an arbitrary constant} \end{aligned}$$

General solution: $\ln|1-2y| = -2x + c$. ■

Remark: Why is it acceptable to leave the answer as $\ln|1-2y| = -2x + c$ *rather than* $\ln|1-2y| = -2x - 2c'$*?*

Removal of modulus/absolute value

$\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$ if $x > 0$ and $\int \frac{1}{x}\,\mathrm{d}x = \ln(-x) + c$ if $x < 0$. This gives us the more general integration formula $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$. Similarly, we typically add an extra modulus to the integration formulas for $\int \frac{1}{x^2 - a^2}\,\mathrm{d}x$ and $\int \frac{1}{a^2 - x^2}\,\mathrm{d}x$ from our formula booklet ListMF26.

The introduction of the modulus simplifies our working during the integration step because we no longer need to check the domain of our function. However, in many situations we often will like to remove it at a later step.

For example, removing the modulus from the previous solution is necessary if we want to make y the subject, which will facilitate further work on our solution (e.g. finding particular solutions and sketching graphs).

We can use the domain of our function to determine whether to take the positive or negative case, but there is another possible approach.

Let's start with our previous solution of $\ln|1 - 2y| = -2x + c$ and remove the logarithm to get $|1 - 2y| = e^{-2x+c}$. Our rules of indices gives us $e^{-2x+c} = e^{-2x}e^{c}$. Note that since c is a constant, e^c is also a constant. Removing the modulus gives us either $1 - 2y = e^c e^{-2x}$ or $1 - 2y = -e^c e^{-2x}$.

We now introduce a new constant A, which we can view as $A = \pm e^c$. This gives us a much simpler expression $1 - 2y = Ae^{-2x}$. We thus convert the problem of domain considerations into a matter of finding the constant A if we're given conditions on x and y.

$$\boxed{\ln|f(y)| = g(x) + c \quad \Longrightarrow \quad f(y) = Ae^{g(x)}}$$

Remark: refer to example 5 for the use of this technique.

For that example, try substituting our initial conditions $t = 0, V = 6.75$ *earlier to find c in* $\frac{\ln|9 - 2V|}{-2} = t + c$. *While that is definitely valid, why does it make our working tougher when we try to make V the subject afterwards?*

10.5 2nd order DEs of the form $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$

Second order DEs of the form $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$ can be solved by integrating twice. Note that there will be two independent integration constants.

Example 4. Find the general solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cos 3x$.

Solution 4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cos 3x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int \cos 3x \,\mathrm{d}x = \frac{\sin 3x}{3} + c$$

$$y = \int \frac{\sin 3x}{3} + c \,\mathrm{d}x$$
$$= -\frac{\cos 3x}{9} + cx + d,$$

where c, d are arbitrary constants

General solution: $y = -\dfrac{\cos 3x}{9} + cx + d.$ ■

10.6 DEs in problem situations

Many problem situations can be translated to differential equations because of the concept of rates of change. For example, consider a quantity x. The rate of change of x is given by $\frac{\mathrm{d}x}{\mathrm{d}t}$ and can be obtained by

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \text{rate of increase of } x - \text{ rate of decrease of } x}$$

Example 5. A water tank with V m^3 of water inside at time t s is being filled up at a rate of 9 m^3/s. At the same time, there is a leak in the tank such that the volume decreases at a rate proportional to the current volume of water inside.

It is given that the volume of water inside the tank remains constant if $V = 4.5$ m^3.

(a) Show that the volume of water inside the tank can be described by $\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - 2V$.

(b) Find V in terms of t, given that the initial volume of water in the tank is 6.75 m^3.

(c) State what happens to the volume of water in the tank eventually.

(d) Sketch the graph of V against t for the parts relevant to the context of the question.

Solution 5.

(a) Rate of increase of V : 9
Rate of decrease of V : kV
$\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - kV$.
When $V = 4.5$, $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.
$0 = 9 - k(4.5) \implies k = 2$.
Hence $\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - 2V$. ■

(b)
$$\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - 2V$$
$$\int \frac{1}{9-2V}\,\mathrm{d}V = \int 1\,\mathrm{d}t$$
$$\frac{\ln|9-2V|}{-2} = t + c$$
$$|9-2V| = e^{-2t-2c}$$
$$9 - 2V = \pm e^{-2c}e^{-2t}$$
$$9 - 2V = Ae^{-2t}, \quad \text{where } c, A \text{ are arbitrary constants}$$

When $t = 0, V = 6.75$.

$9 - 2(6.75) = Ae^0 \implies A = -\frac{9}{2}$

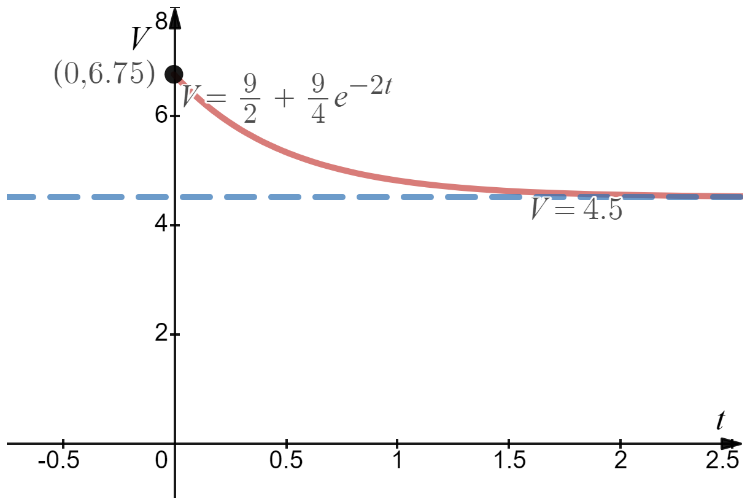
$9 - 2V = -\dfrac{9}{2}e^{-2t}$

$V = \dfrac{9}{2} + \dfrac{9}{4}e^{-2t}$. ■

(c) As $t \to \infty$, $e^{-2t} \to 0$. Hence $V \to \frac{9}{2}$.

The volume of water in the tank approaches 4.5 m^3 eventually. ■

(d) The parts of the graph relevant to the context of the question are when $t > 0, V > 0$.



■

Credits: graph created with Desmos:
www.desmos.com/calculator/pg4oo6j5yn

10.7 Solving DEs by substitution

More complicated DEs can sometimes be reduced to the forms we have encountered in earlier sections via a substitution. Solving DEs by substitution can be carried out in four broad steps outlined below. For illustration, we will assume we have an original DE with variables x and y and have a substitution relating a new variable z to them.

- Step 1: Differentiate the given substitution implicitly.
  (e.g. Find $\frac{\mathrm{d}z}{\mathrm{d}x}$ in terms of $\frac{\mathrm{d}y}{\mathrm{d}x}, x, y$ and z)
- Step 2: Replace the old variables and derivatives in the original DE.
  (e.g. Replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y with $\frac{\mathrm{d}z}{\mathrm{d}x}$ and z using the substitution)
- Step 3: Solve the new DE (which should be simpler).
  (e.g. Find a relation between z and x by integration)
- Step 4: Replace the new variable back with the original one.
  (e.g. Replace z to find a solution involving just x and y)

Example 6. Use the substitution $y = zx^2$ to find the general solution of $x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - 1$.

Solution 6. Differentiating $y = zx^2$ implicitly w.r.t. x (by product rule),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}x^2 + 2zx \tag{1}$$

Substituting (1) and $y = zx^2$ into $x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - 1$,

$$\begin{aligned} x\left(\frac{\mathrm{d}z}{\mathrm{d}x}x^2 + 2zx\right) &= 2zx^2 - 1 \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= -\frac{1}{x^3} \\ z &= \int -\frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2x^2} + c \end{aligned}$$

Substituting $z = \frac{y}{x^2}$, $\quad \frac{y}{x^2} = \frac{1}{x^2} + c$.

General solution: $y = \frac{1}{2} + cx^2$. ■

Links and other resources

- Online version of these notes (with less explanation text) is available at math-atlas.vercel.app/notes/de
- Computer generated questions: math-atlas.vercel.app/questions
- YouTube channel with worked TYS solutions and revision lectures http://tiny.cc/kelvinsoh
- Contact me at kelvinsohmath@gmail.com