10. Differential equations

Theory

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- **Differential equations** are equations involving variables (e.g. x, y) and their derivatives $\left(\text{e.g. } \frac{dy}{dx}\right)$.
- They can be solved (i.e. finding an equation between x and y only) by integration.
- A differential equation is of **variable separable** form if it can be written as

$$g(y)\frac{dy}{dx} = f(x).$$

- Variable separable equations can be solved by proceeding to integrate:

$$\int g(y)\, dy = \int f(x)\, dx.$$

- Differential equations of the form $\frac{d^2y}{dx^2} = f(x)$ can be solved by integrating twice.
- The **general solution** refers to all possible solutions of a differential equation. There are infinite number of solutions since the integration constant C can take any value.
- A **particular solution** refers to one solution of a differential equation. Typically a set of values will be given to us to find C in order to obtain a particular solution.

Problem sums: rates of change

Rate of change of x: $\frac{dx}{dt}$

$$\boxed{\frac{dx}{dt} = \text{rate of increase of } x - \text{rate of decrease of } x}$$

Second order DE

$$\frac{d^2y}{dx^2} = 2x$$
$$\frac{dy}{dx} = x^2 + C$$
$$y = \frac{x^3}{3} + Cx + D.$$

Removing the modulus

$$\ln|x| = t + C$$
$$|x| = e^{t+C}$$
$$|x| = Be^t \qquad \text{where } B = e^C$$
$$x = Ae^t \qquad \text{where } A = \pm B$$

Substitution

Approach

- Step 1: Differentiate given substitution implicitly.
- Step 2: Using the given substitution and the differentials in step 1, replace the old variable (e.g. y) to the new one (e.g. u).
- Step 3: The DE in the new variable should be of separable form. Integrate it.
- Step 4: Substitute back the old variable.

Variable separable form

$$\frac{dy}{dx} = xy^2$$
$$\frac{1}{y^2}\frac{dy}{dx} = x$$
$$\int \frac{1}{y^2}\, dy = \int x\, dx$$

General solution: $-\frac{1}{y} = \frac{x^2}{2} + C.$

If $y = 1$ when $x = 0$, $C = -1$

Particular solution: $-\frac{1}{y} = \frac{x^2}{2} - 1.$

Example: substitution method

Example

Use the substitution $u = xy$ to solve

$$x^2y\frac{dy}{dx} + xy^2 = 1$$

Differentiating $u = xy$ implicitly with respect to x,
$$\frac{du}{dx} = x\frac{dy}{dx} + y$$

Hence $x\frac{dy}{dx} = \frac{du}{dx} - y$

Substituting into given DE,
$$xy\left(\frac{du}{dx} - y\right) + uy = 1$$

$$u\frac{du}{dx} = 1$$
$$\frac{u^2}{2} = x + C$$
$$u^2 = 2x + C'$$

Solution: $(xy)^2 = 2x + C'$