10 Differential equations

10.1 Introduction and definitions

A **differential equation (DE)** is a relation between variables (e.g. x, y) and their derivatives (e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}$). For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ is a DE.

A **solution** to a DE is a relation between our variables (e.g. x, y) that satisfy the DE. For example, $y = x^2$ is a solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

The **general solution** refers to a solution that includes all the possible solutions of a DE. For $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, the general solution is $y = x^2 + c$, where c is an arbitrary constant.

A **particular solution** refers to one specific solution of a DE. For example, for $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, if we are further given that $y = 3$ when $x = 0$, then $y = x^2 + 3$ is the particular solution.

10.2 Verifying solutions

Example 1. Verify that $y = x^3 e^{x^2}$ is a solution of the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} = y(3 + 2x^2)$.

Solution 1. Differentiating $y = x^3 e^{x^2}$ w.r.t. x (using the product rule),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 e^{x^2} + x^3 \left(2xe^{x^2}\right)$$

Multiplying x on both sides,

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^3 e^{x^2} + 2x^2 \left(x^3 e^{x^2}\right)$$

By observing that $y = x^3 e^{x^2}$, we have

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3y + 2x^2 y \\ &= y(3 + 2x^2) \end{aligned}$$

This verifies that $y = x^3 e^{x^2}$ is a solution. ■

10.3 Solving DEs of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \quad \Longrightarrow \quad y = \int f(x)\,\mathrm{d}x}$$

Example 2. Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x^2+1}$.

(a) Find the general solution.

(b) Find the particular solution, given that $y = 2$ when $x = 0$.

Solution 2.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{x^2+1}$

$$y = \int \frac{1}{x^2+1}\,\mathrm{d}x$$

$$y = \tan^{-1} x + c$$

General solution: $y = \tan^{-1} x + c$, where c is an arbitrary constant. ■

(b) When $x = 0$, $y = 2$

$2 = \tan^{-1} 0 + c$

$c = 2$

Particular solution: $y = \tan^{-1} x + 2$. ■

10.4 Solving DEs of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(y)$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(y) \quad \implies \quad \int \frac{1}{f(y)}\,\mathrm{d}y = \int 1\,\mathrm{d}x}$$

Example 3. Find the general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2y$.

Solution 3.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - 2y \\ \int \frac{1}{1-2y}\,\mathrm{d}y &= \int 1\,\mathrm{d}x \\ \frac{\ln|1-2y|}{-2} &= x + c', \text{ where } c' \text{ is an arbitrary constant}\end{aligned}$$

General solution: $\ln|1-2y| = -2x + c$. ■

Removal of modulus/absolute value

$$\boxed{\ln|f(y)| = g(x) + c \quad \implies \quad f(y) = Ae^{g(x)}}$$

10.5 2nd order DEs of the form $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$

Example 4. Find the general solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cos 3x$.

Solution 4.

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \cos 3x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \int \cos 3x\,\mathrm{d}x = \frac{\sin 3x}{3} + c\end{aligned}$$

$$\begin{aligned}y &= \int \frac{\sin 3x}{3} + c\,\mathrm{d}x \\ &= -\frac{\cos 3x}{9} + cx + d,\\ &\text{where } c, d \text{ are arbitrary constants}\end{aligned}$$

General solution: $y = -\dfrac{\cos 3x}{9} + cx + d$. ■

10.6 DEs in problem situations

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \text{rate of increase of } x - \text{ rate of decrease of } x}$$

Example 5. A water tank with V m^3 of water inside at time t s is being filled up at a rate of 9 m^3/s. At the same time, there is a leak in the tank such that the volume decreases at a rate proportional to the current volume of water inside.

It is given that the volume of water inside the tank remains constant if $V = 4.5$ m^3.

(a) Show that the volume of water inside the tank can be described by $\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - 2V$.

(b) Find V in terms of t, given that the initial volume of water in the tank is 6.75 m^3.

(c) State what happens to the volume of water in the tank eventually.

(d) Sketch the graph of V against t for the parts relevant to the context of the question.

Solution 5.

(a) Rate of increase of V : 9

Rate of decrease of V : kV

$\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - kV$.

When $V = 4.5$, $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.

$0 = 9 - k(4.5) \implies k = 2$.

Hence $\frac{\mathrm{d}V}{\mathrm{d}t} = 9 - 2V$. ■

(b)
$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 9-2V\\
\int \frac{1}{9-2V}\,\mathrm{d}V &= \int 1\,\mathrm{d}t\\
\frac{\ln|9-2V|}{-2} &= t+c\\
|9-2V| &= e^{-2t-2c}\\
9-2V &= \pm e^{-2c}e^{-2t}\\
9-2V &= Ae^{-2t}, \qquad \text{where } c, A \text{ are arbitrary constants}
\end{aligned}$$

When $t=0, V=6.75$.

$9-2(6.75) = Ae^0 \implies A = -\frac{9}{2}$

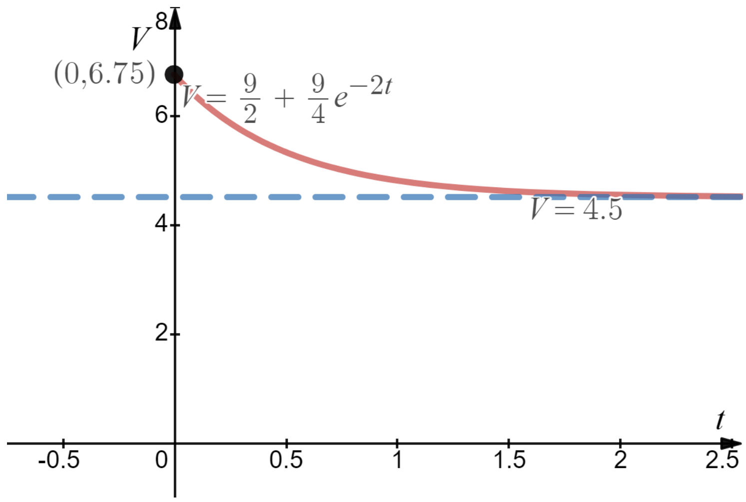
$9-2V = -\frac{9}{2}e^{-2t}$

$V = \frac{9}{2} + \frac{9}{4}e^{-2t}$. ■

(c) As $t \to \infty$, $e^{-2t} \to 0$. Hence $V \to \frac{9}{2}$.

The volume of water in the tank approaches 4.5 m^3 eventually. ■

(d) The parts of the graph relevant to the context of the question are when $t > 0, V > 0$.



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Credits: graph created with Desmos:
www.desmos.com/calculator/pg4oo6j5yn

10.7 Solving DEs by substitution

- Step 1: Differentiate the given substitution implicitly.
  (e.g. Find $\frac{\mathrm{d}z}{\mathrm{d}x}$ in terms of $\frac{\mathrm{d}y}{\mathrm{d}x}, x, y$ and z)
- Step 2: Replace the old variables and derivatives in the original DE.
  (e.g. Replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y with $\frac{\mathrm{d}z}{\mathrm{d}x}$ and z using the substitution)
- Step 3: Solve the new DE (which should be simpler).
  (e.g. Find a relation between z and x by integration)
- Step 4: Replace the new variable back with the original one.
  (e.g. Replace z to find a solution involving just x and y)

Example 6. Use the substitution $y = zx^2$ to find the general solution of $x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - 1$.

Solution 6. Differentiating $y = zx^2$ implicitly w.r.t. x (by product rule),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}x^2 + 2zx \tag{1}$$

Substituting (1) and $y = zx^2$ into $x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - 1$,

$$\begin{aligned} x\left(\frac{\mathrm{d}z}{\mathrm{d}x}x^2 + 2zx\right) &= 2zx^2 - 1 \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= -\frac{1}{x^3} \\ z &= \int -\frac{1}{x^3}\,\mathrm{d}x = \frac{1}{2x^2} + c \end{aligned}$$

Substituting $z = \frac{y}{x^2}$, $\quad \frac{y}{x^2} = \frac{1}{x^2} + c$.

General solution: $y = \frac{1}{2} + cx^2$. ■