# 13 Complex numbers

## Introduction: extensions of number systems

Our first mathematical encounter is likely with the natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$. We can use them to count, and they are well suited for the operations of addition and multiplication. Given any two natural numbers, the sum or product of them will be a natural number too.

Subtraction, unfortunately, is not so well defined at the moment; we have to subtract a smaller number from a larger number in order to stay in the natural numbers. To overcome this problem we can define negative numbers, and together they form the integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. We say that the integers are an extension of the natural numbers because the natural numbers still keep all the important properties from before but the inclusion of the new negative numbers allow for more to be done.

In a similar vein, the problem of dividing numbers which leave remainders lead us to extend the integers to the rational numbers $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$.

The study of geometry and change often leads us to the fundamental constants $\pi$ and $e$, both of which are irrational. Square roots are also limited to the perfect squares. Thus, we extend the rational numbers and often work in the real numbers, $\mathbb{R}$, in our study. It is instructive to visualize them on a number line, and now $\sqrt{x}$ exists as long as $x$ is non-negative.

Complex numbers, $\mathbb{C}$, came about from imagining if square roots of negative numbers are allowed. And despite considerable resistance to the idea when it was first introduced, they allowed for breakthroughs in the solving of cubic equations. Decades down the road this seemingly theoretical and abstract construction has also found applications in the study of real world phenomena such as in electrical engineering and quantum mechanics.

Square roots of negative numbers do not exist in the real number system. The idea of the complex number is to introduce a new number, $i$, thought of as $\sqrt{-1}$. Every new number (called complex) is then in the form $x + yi$, where $x, y \in \mathbb{R}$. This extends our real numbers because every real number, for example, $\sqrt{2}$, is also a complex number of the form $\sqrt{2} + 0i$.

## 13.1 Cartesian form arithmetic

### Terminology

We will often denote complex numbers with the letters $z$ and $w$, and write them in the forms $z = x + yi$ and $w = a + bi$. This is called the **cartesian form.**

$x$ is then called the **real part** of $z$, denoted $\text{Re}(z)$, while $y$ is called the **imaginary part** of $z$, denoted $\text{Im}(z)$.

For example, for the complex number $z = 2 - 3i$, $\text{Re}(z) = 2$ and $\text{Im}(z) = -3$.

*Remark: Im(z)* $= -3$ *and Im*$(z) \neq -3i$.

### Powers of $i$

Because $i$ can be thought of as $\sqrt{-1}$, we have $\boxed{i^2 = -1}$. Then $i^3 = i^2, i = -i$ and $i^4 = (i^2)^2 = 1$.

*Concept check 1: what is* $i^{2021}$*?*

### Addition and subtraction

Addition and subtraction of complex numbers is pretty intuitive: we add the real and imaginary parts accordingly. This behaves just like treating $i$ as our usual algebraic unknown. For example,

$$(2 - i) + (3 + 4i) = 5 + 3i$$
$$(1 + 2i) - (3 - 5i) = -2 + 7i$$

### Multiplication

Multiplication of complex numbers can be thought of as our usual algebraic expansion, except that $i^2 = -1$ allows us to further simplify our answer. For example,

$$\begin{aligned}(2 + i)(3 + 4i) &= 6 + 8i + 3i + 4i^2 \\ &= 6 + 11i - 4 \\ &= 2 + 11i\end{aligned}$$

## Complex conjugates

Similar to some of our previous work on surds, the idea of the conjugate helps in the manipulation of complex numbers. Given a complex number $z = x+yi$, its **complex conjugate**, denoted by $z^*$, is given by $\boxed{z^* = x - yi}$, where we switch the sign of the imaginary part.

The gives rise to a few useful formulas:

$$z + z^* = 2x = 2\text{Re}(z) \tag{1}$$

$$z - z^* = 2yi = 2\text{Im}(z)i \tag{2}$$

$$\boxed{zz^* = x^2 + y^2} \tag{3}$$

In particular, formula (3) will be used in the next section on complex division.

*Concept check 2: derive the 3 formulas above.*
*Concept check 3: what can we say about $z$ if $z = z^*$?*

## Division

To carry out complex division, we make use of the complex conjugate in a way similar to how we "rationalize surds". Formula (3) in the previous section then simplifies the denominator into a real number, enabling the final result to be back into our cartesian form $x + yi$.

$$\begin{aligned}\frac{2+i}{3-4i} &= \frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} \\ &= \frac{2+11i}{3^2+4^2} \\ &= \frac{2+11i}{25} \\ &= \frac{2}{25} + \frac{11}{25}i\end{aligned}$$

## 13.2 Solving equations

### Linear equations

In the real numbers, to solve a linear equation like $3x - 5 = 4$, we move terms around until $x$ is the subject of the equation. This can also be done when handling linear equations involving complex numbers.

**Example 1.** Solve the equation $(2 + 3i)z + 8 + 2i = 1 - i$.

**Solution 1.**

$$\begin{aligned}(2+3i)z &= 1 - i - (8 + 2i)\\(2+3i)z &= -7 - 3i\\z &= \frac{-7-3i}{2+3i}\\&= \frac{-7-3i}{2+3i} \times \frac{2-3i}{2-3i}\\&= \frac{-23+15i}{13}. \quad \blacksquare\end{aligned}$$

### Quadratic equations

In the real numbers, the discriminant $b^2 - 4ac$ gives us a condition on the number of real roots a quadratic equation has. In complex numbers, we can now take square roots of negative numbers so all quadratic equations will have 2 complex roots (including multiplicity). The quadratic formula we know still applies and is especially useful.

**Example 2.** Solve the equation $2z^2 - 2x + 5 = 0$.

**Solution 2.**

$$\begin{aligned}z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(5)}}{2(2)}\\&= \frac{2 \pm \sqrt{-36}}{4}\\&= \frac{2 \pm 6i}{4}\\&= \frac{1 \pm 3i}{2}. \quad \blacksquare\end{aligned}$$

## Comparing real and imaginary parts

For two complex numbers to be equal, both the real and imaginary parts must be the same. Thus one complex equation can actually help us solve for two real unknowns.

*Remark: Comparison only works if all our variables are real. For example, we cannot do comparison for the equation $z+2zi = 3+4i$ where $z$ is complex (why not?). On the other hand, if $x, y \in \mathbb{R}$, we can compare for the equation $x + 2yi = 3 + 4i$ to get $x = 3, y = 2$.*

When the strategies outlined in examples 1 and 2 do not work, we can make use of the comparison technique. Since it only works for real unknowns, we can let $z = x + yi$, where $x$ and $y$ are real.

**Example 3.** Solve the equation $-18iz + zz^* = -56 - 90i$.

**Solution 3.** Let $z = x + yi, \quad x, y \in \mathbb{R}$.

$$\begin{aligned} -18i(x + yi) + (x + yi)(x - yi) &= -56 - 90i \\ -18xi + 18y + x^2 + y^2 &= -56 - 90i \\ (y^2 + 18y + x^2) - 18xi &= -56 - 90i \end{aligned}$$

Comparing imaginary parts, $-18x = -90$ so $x = 5$.
Substituting $x = 5$ and comparing real parts,

$$\begin{aligned} y^2 + 18y + 25 &= -56 \\ y^2 + 18y + 81 &= 0 \\ (y + 9)^2 &= 0 \\ y &= -9 \end{aligned}$$

Hence $z = 5 - 9i$. ■

## 13.3 The conjugate root theorem

### Two additional theorems

In our previous study we may have observed an interesting phenomena: that linear equations (highest power of 1) tend to have one solution, quadratics tend to have two, cubic equations tend to have three and so on. That was not always the case when working with real numbers, but working with complex numbers give us the following beautiful result:

**Theorem** (Fundamental Theorem of Algebra)**.** *The polynomial equation* $az^n + bz^{n-1} + \ldots = 0, a \neq 0$ *has exactly* $n$ *complex roots (including multiplicity).*

One approach to solve polynomial equations uses the factor theorem:

**Theorem** (Factor Theorem)**.** *If* $z = \alpha$ *is a root of a polynomial* $f(z)$ *(i.e.* $f(\alpha) = 0$*), then* $(z - \alpha)$ *is a factor of* $f(z)$*.*

### The conjugate root theorem

The factor theorem allows us to break polynomials up one root/factor at a time. This can be quite tedious when solving polynomial equations of large order. Working with complex numbers can potentially speed things up via the **conjugate root theorem**.

**Theorem** (Conjugate Root Theorem)**.** *Let* $P(z)$ *be a polynomial with* ***real coefficients****. If* $z = a + bi$ *is a root of* $P(z) = 0$*, then its complex conjugate* $z^* = a - bi$ *is also a root.*

*Remark: We can apply the conjugate root theorem to a polynomial like* $z^2 - z + 3$ *because all the coefficients are real, even though the roots are complex. Meanwhile, the conjugate root theorem does not apply to a polynomial like* $z^2 - 2iz + 3 + 4i$*.*

## Expansion of factors from conjugate roots

The following is a derivation of a useful identity to obtain quadratic factors. In the first case, we make use of the $(a+b)(a-b) = a^2 - b^2$ formula.

$$\begin{aligned}(z - a + bi)(z - a - bi) &= (z-a)^2 - (bi)^2 \\ &= z^2 - 2a + a^2 + b^2\end{aligned}$$

We will be working with complex numbers in polar form in section 13.4. The following is a similar derivation of the above result in polar form making use of the $z + z^* = 2r\cos(\theta)$ formula.

$$\begin{aligned}(z - re^{i\theta})(z - re^{-i\theta}) &= z^2 - re^{i\theta}z - re^{-i\theta}z + re^{i\theta}re^{-i\theta} \\ &= z^2 - rz(re^{i\theta} + re^{-i\theta}) + r^2 \\ &= z^2 - (2r\cos\theta)z + r^2\end{aligned}$$

## Solving polynomial equations cubic and above

We will illustrate the use of the conjugate root theorem in solving polynomial equations with an example.

**Example 4.** Consider the equation $2z^3 - z^2 + 14z + 30 = 0$.

(a) Verify that $1 + 3i$ is a root of the equation.

(b) Hence solve the equation.

**Solution 4.**
(a) We first evaluate $(1+3i)^2$ and $(1+3i)^3$.

$$\begin{aligned}(1+3i)^2 &= 1 + 6i + 9i^2 \\ &= -8 + 6i \\ (1+3i)^3 &= (1+3i)(-8+6i) \\ &= -8 + 18i^2 - 24i + 6i \\ &= -26 - 18i\end{aligned}$$

Substituting $z = 1 + 3i$ into the equation

$$\begin{aligned}\text{LHS} &= 2(-26 - 18i) - (-8 + 6i) + 14(1 + 3i) + 30 \\ &= (-52 - 36i) + (8 - 6i) + (14 + 42i) + 30 \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

Hence $1 + 3i$ is a root of the equation. ■

(b) Since the equation has only real coefficients, by the conjugate root theorem, $1-3i$ is also a root.

By the factor theorem, both $z-(1+3i)=(z-1-3i)$ and $(z-1+3i)$ are factors of $2z^3-z^2+14z+30$.

Expanding the two factors:

$$(z-1-3i)(z-1+3i)=z^2-2z+10$$

We can then obtain the remaining factor either by **long division** or **comparing coefficients**.

$$\begin{array}{r|l} & 2z + 3 \\ z^2-2z+10 & 2z^3 - z^2 + 14z + 30 \\ & -\,2z^3 + 4z^2 - 20z \\ & 3z^2 - 6z + 30 \\ & -\,3z^2 + 6z - 30 \\ & 0 \end{array}$$

Let $2z^3-z^2+14z+30$
$=(z^2-2z+10)(az+b)$
Comparing coefficients:

$z^3: \quad a=2$

$z^0: \quad 10b=30$

$\qquad b=3$

Hence the final factor is $(2z+3)$.

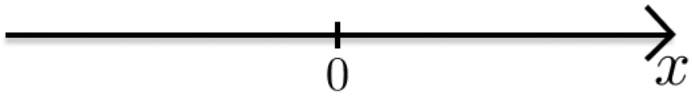
$(z-1-3i)(z-1+3i)(2z+3)=0.$

$z=1+3i,\ z=1-3i$ or $z=-\frac{3}{2}$. ■

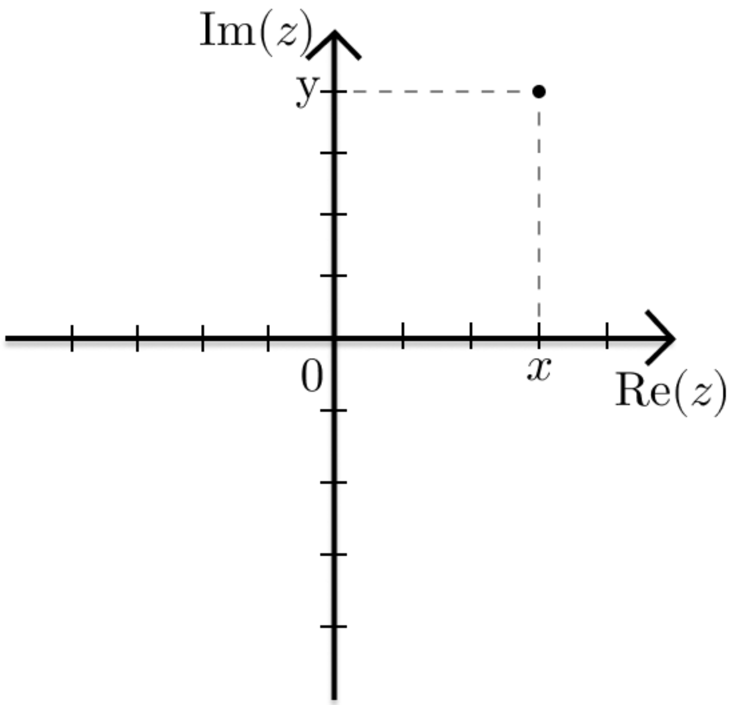
## 13.4 Argand diagram and polar form

### The Argand diagram

A useful visualization of the real numbers is the number line: the number 0 is in the middle, and the line extends infinitely in both directions.



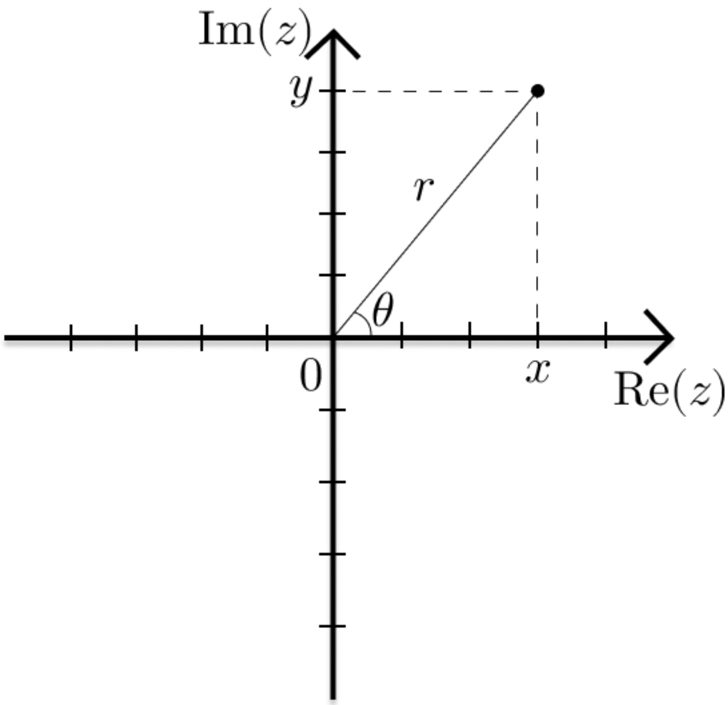

We can think of the complex numbers as extending the real number line into a complex number plane. We then can place each complex number with the real part on the $x$-axis and the imaginary part on the $y$-axis. For example, the complex number $3 + 4i$ can be represented by the point $(3, 4)$ on the complex plane. We call such a diagram an Argand diagram.



## The polar (modulus-argument) form

So far we have worked with the cartesian form $x + yi$ of a complex number. On an Argand diagram that corresponds to the $x$- and $y$-coordinates.

Another way to understand complex numbers is called the **polar** (or **modulus-argument**) form. Instead of focusing on the $x$- and $y$-coordinates, we focus on two quantities called the **modulus**, denoted by $|z|$ or $r$, and the **argument**, denoted by $\arg(z)$ or $\theta$.




As seen on the diagram above, the modulus, $r$, refers to the distance from the point representing the complex number to the origin. Meanwhile, the argument, $\theta$, refers to anticlockwise angle the positive $x$-axis makes with the line from the origin to the point.

It turns out that working with the polar form greatly simplifies complex multiplication, division and exponentiation (powers) and leads to many useful and beautiful results. But before that, we will first look at the three forms of a complex number and learn how to convert from one to another.

**Three forms of a complex number**

- Cartesian form $\boxed{z = x + yi.}$
- Trigonometric form $\boxed{z = r(\cos\theta + i\sin\theta).}$
- Exponential form $\boxed{z = re^{i\theta}.}$

The figure in the previous page shows that $x, y$ and $r, \theta$ are related by a right angle triangle, giving rise to the following formulas:

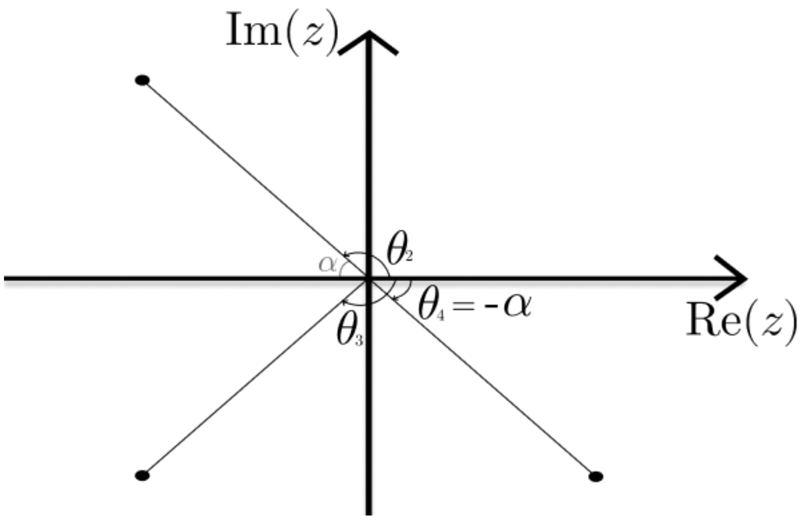
**Converting cartesian form to polar form**

To calculate $|z| = r$ from $x$ and $y$, $\boxed{|z| = r = \sqrt{x^2 + y^2}.}$

Meanwhile, $\arg(z) = \theta$ is related to $x$ and $y$ by $\tan\theta = \frac{y}{x}$.

Finding $\theta$ is more complicated than simply taking $tan^{-1}$, however, due to the many solutions of the equation from the different quadrants. Thus, we take a two step approach in finding the argument. We first find the basic angle, $\alpha$, and consider the signs of $x$ and $y$ to determine $\theta$.

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right|, \qquad \arg(z) = \theta = \begin{cases} \alpha & \text{if } x > 0, y > 0 \\ \pi - \alpha & \text{if } x < 0, y > 0 \\ -(\pi - \alpha) & \text{if } x < 0, y < 0 \\ -\alpha & \text{if } x > 0, y < 0 \end{cases}$$



**Example 5.** (A) Convert $z = -1 + i$ into polar form.

**Solution 5.** $r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$.

$\alpha = \tan^{-1} |-1| = \frac{\pi}{4}$.

Since we are in the second quadrant, $\theta = \pi - \alpha = \frac{3\pi}{4}$.

Hence $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}e^{i\frac{3\pi}{4}}$. ■

**Converting polar form to cartesian form**

To calculate $x$ and $y$ from $r$ and $\theta$ instead, we note that $x$ is the adjacent side of our right angle triangle and $y$ is the opposite side. Trigonometry then gives us the formulas $\boxed{x = r\cos\theta}$ and $\boxed{y = r\sin\theta.}$

In fact, we can actually view the trigonometric form of a complex number $z = r(\cos\theta + i\sin\theta)$ as an intermediate between the exponential form $z = re^{i\theta}$ and the cartesian form $z = x + yi$. Comparing the real and imaginary parts of the cartesian and trigonometric form leads us to the formulas above.

**Example 5.** (B) Convert $z = 2e^{i\frac{2\pi}{3}}$ into polar form.

**Solution 5.**

$$\begin{aligned}
2e^{i\frac{2\pi}{3}} &= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\
&= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \\
&= -1 + \sqrt{3}i. \quad ■
\end{aligned}$$

# 13.5 Polar form arithmetic

In working with complex numbers in cartesian form, we may have noticed that while addition and subtraction is pretty straightforward, the mechanics of multiplication, division and exponentiation (taking powers) can get pretty tedious to execute. It turns out that using the polar form greatly simplifies these operations.

## Arithmetic in exponential form

Given our previous work with indices and exponentials, working with complex numbers in exponential form should feel natural and very similar. Meanwhile, complex conjugates are reflections of each other about the $\mathrm{Re}(z)$-axis on the Argand diagram. This means that the modulus stays the same but the sign of the argument is flipped.

$$\begin{aligned} wz &= \left(r_1 e^{i\theta_1}\right) \cdot \left(r_1 e^{i\theta_2}\right) = (r_1 r_2) e^{i(\theta_1+\theta_2)} \\ \frac{w}{z} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \\ z^n &= \left(re^{i\theta}\right)^n = r^n e^{in\theta} \\ z^* &= \left(re^{i\theta}\right)^* = re^{-i\theta} \end{aligned}$$

## Modulus and argument formulas

The formulas above can be broken down into formulas relating to just the modulus or the argument themselves.

$$\begin{aligned} |wz| &= |w|\,|z| \\ \left|\frac{w}{z}\right| &= \frac{|w|}{|z|} \\ |z^n| &= |z|^n \\ |z^*| &= |z| \end{aligned}$$

$$\begin{aligned} \arg(wz) &= \arg(w) + (z) \\ \arg\left(\frac{w}{z}\right) &= \arg(w) - (z) \\ \arg(z^n) &= n\arg(z) \\ \arg(z^*) &= -\arg(z) \end{aligned}$$

## Principal values

The problem when working with angles/arguments is that multiple values can refer to the same angle when we go extra rounds (for example, visualize the angle described by $\frac{\pi}{2}$ and $\frac{5\pi}{2}$). It is thus useful to decide what values are considered the "simplest" form we should leave our answers in.

For our purposes we will typically leave our arguments in the principal range $-\pi < \theta \leq \pi$. Any values outside this range can be converted by adding/subtraction $2k\pi$ ($k$ extra rounds), where $k \in \mathbb{Z}$. This can be seen in the last two steps of the following example.

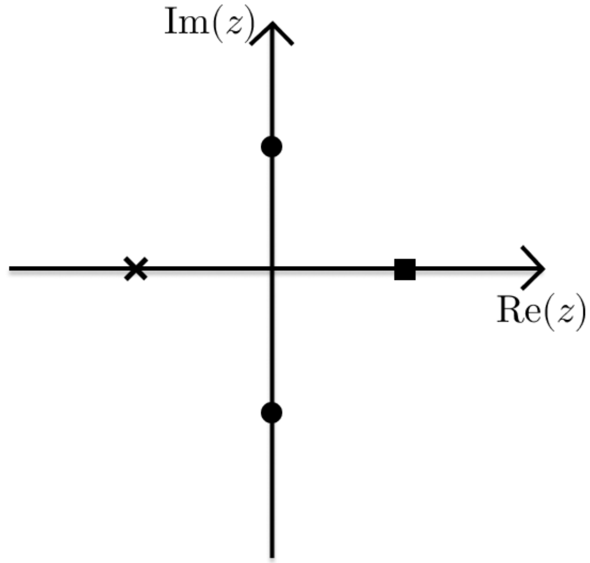
**Example 6.** Evaluate $\left(\frac{\left(e^{-\frac{\pi}{12}i}\right)^7 e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^*$.

**Solution 6.**

$$\begin{aligned}
\left(\frac{\left(e^{-\frac{\pi}{12}i}\right)^7 e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^* &= \left(\frac{e^{-\frac{7\pi}{12}i}e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^* \\
&= \left(\frac{e^{-\frac{7\pi}{12}i+\left(-\frac{5\pi}{6}i\right)}}{e^{\frac{\pi}{3}i}}\right)^* \\
&= \left(\frac{e^{-\frac{17\pi}{12}i}}{e^{\frac{\pi}{3}i}}\right)^* \\
&= \left(e^{-\frac{17\pi}{12}i-\frac{\pi}{3}i}\right)^* \\
&= \left(e^{-\frac{7\pi}{4}i}\right)^* \\
&= e^{\frac{7\pi}{4}i} \\
&= e^{\left(\frac{7\pi}{4}-2\pi\right)i} \\
&= e^{-\frac{\pi}{4}i} \quad \blacksquare
\end{aligned}$$

## 13.6 Real and purely imaginary numbers

Complex numbers have two parts: the real part and the imaginary part. A number is real if its imaginary part is 0. Meanwhile, a number is purely imaginary if its real part is 0. These numbers can be located on an Argand diagram as shown below.




The following table lists out the conditions for a number to be real/purely imaginary. The extra $k, k \in \mathbb{Z}$ in the conditions account for multiple answers due to arguments outside the principal range. $k\pi$ refers to $k$ extra half-rounds while $2k\pi$ refers to $k$ extra full rounds around the Argand diagram.

| Condition | Cartesian form, $x + yi$ | Polar form, $re^{i\theta}$ |
|---|---|---|
| Real | $y = 0$ | $\theta = k\pi, \quad k \in \mathbb{Z}$ |
| Real and positive | $y = 0, x > 0$ | $\theta = 2k\pi, \quad k \in \mathbb{Z}$ |
| Real and negative | $y = 0, x < 0$ | $\theta = \pi + k\pi, \quad k \in \mathbb{Z}$ |
| Purely imaginary | $x = 0$ | $\theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$ |

**Example 7.** Find the three smallest positive integers $n$ such that $\left(e^{-\frac{\pi}{12}i}\right)^n$ is purely imaginary.

**Solution 7.**

$$\arg\left(\left(e^{-\frac{\pi}{12}i}\right)^n\right) = n\arg\left(e^{-\frac{\pi}{12}i}\right) = -\frac{n\pi}{12}.$$

For it to be purely imaginary,

$$\begin{aligned} -\frac{n\pi}{12} &= \frac{\pi}{2} + k\pi \\ -n\pi &= 6\pi + 12k\pi \\ n &= -6 - 12k \end{aligned}$$

The smallest positive integers $n$ correspond to when $k = -1, -2, -3$.

Hence the three smallest positive integers $n = 6, 18, 24$. ■

# 13.7 Miscellaneous examples, techniques

## A consolidated example

The following is an example combining some of the techniques discussed in this chapter.

**Example 8.** The complex number $z$ is given by $z = 3 + bi$, where $b$ is a real number.

(a) Find the possible values of $b$ if $\frac{z^2}{z^*}$ is real.

For the rest of the question, it is further given that $b > 0$.

(b) Find the smallest integer value of $n$ such that $|z^n| > 1000$.

(c) For the value of $n$ found in (b), find the values of $|z^n|$ and $\arg(z^n)$ such that $-\pi < \arg(z^n) \leq \pi$.

(d) On a single Argand diagram mark out the points $A, B, D, D$ and $E$ representing the complex numbers $z, \frac{z^2}{z}, z^*, \frac{18}{z}$ and $\frac{z^2}{6}$ respectively.

**Solution 8.**
(a) We first evaluate $\frac{z^2}{z^*}$ in terms of $b$.

$$\begin{aligned}\frac{z^2}{z^*} &= \frac{(3+bi)^2}{3-bi} \\ &= \frac{9-b^2+6bi}{3-bi}\cdot\frac{3+bi}{3+bi} \\ &= \frac{3(9-b^2)-6b^2+((9-b^2)b+18b)\,i}{9+b^2}\end{aligned}$$

Since $\frac{z^2}{z^*}$ is real, its imaginary part is 0.

$$\begin{aligned}(9-b^2)b+18b &= 0 \\ b\left(9-b^2+18\right) &= 0 \\ b(27-b^2) &= 0\end{aligned}$$

Hence $b=0$ or $b=\pm\sqrt{27}=\pm 3\sqrt{3}$. ■

(b) Since $b>0, b=3\sqrt{3}$.

$$\begin{aligned}|z^n| &> 1000 \\ \left|3+3\sqrt{3}i\right|^n &> 1000 \\ \left(\sqrt{3^2+(3\sqrt{3})^2}\right)^n &> 1000 \\ 6^n &> 1000 \\ n\ln 6 &> \ln 1000 \\ n &> 3.8553\end{aligned}$$

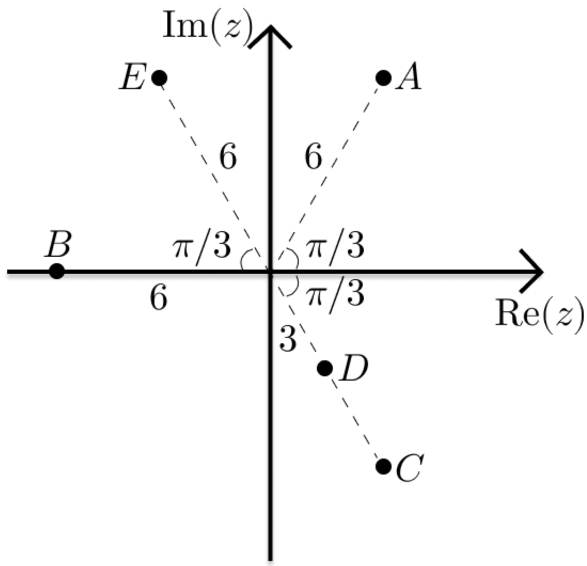
Hence the smallest integer value of $n=4$. ■

(c) For $z=3+3\sqrt{3}i$ and $n=4$,

$$\begin{aligned}|z^n| &= |z|^n \\ &= 6^4 \\ &= 1296.\end{aligned}$$ ■

$$\begin{aligned}\arg(z) &= \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right)\\ &= \frac{\pi}{3}\\ \arg(z^n) &= n\arg(z)\\ &\equiv 4\left(\frac{\pi}{3}\right) - 2\pi\\ &= -\frac{2\pi}{3}. \quad \blacksquare\end{aligned}$$

(d) $z = 6e^{i\frac{\pi}{3}}$, $\quad \dfrac{z^2}{z^*} = \dfrac{36e^{i\frac{2\pi}{3}}}{6e^{-i\frac{\pi}{3}}} = 6e^{i\pi}$, $\quad z^* = 6e^{-i\frac{\pi}{3}}$,
$\dfrac{18}{z} = \dfrac{18}{6e^{i\frac{\pi}{3}}} = 3e^{-i\frac{\pi}{3}}$, $\quad \dfrac{z^2}{6} = \dfrac{36e^{i\frac{2\pi}{3}}}{6} = 6e^{i\frac{2\pi}{3}}$.




## Conjugate formulas in polar form

Now that we have learned about the polar form, we can update formulas (1) to (3) in section 13.1 to reflect polar notation.

$$z + z^* = 2x = 2\text{Re}(z) = 2\cos\theta \tag{1*}$$
$$z - z^* = 2yi = 2\text{Im}(z)i = 2i\sin\theta \tag{2*}$$
$$\boxed{zz^* = x^2 + y^2 = |z|^2} \tag{3*}$$

## The half-angle "trick"

In section section 13.5 we have seen that the polar form works exceedingly well when handling multiplication, division and exponentiation. However, unlike the cartesian form, addition and subtraction is not straightforward to carry out at all and any general approach will probably have to be done using the trigonometric form and trigonometric formulas.

For the special case of adding/subtracting by 1 (after factoring out $|z|$), there is a trick that can be used to greatly simplify the process. There are two key "tricks" we will apply: (a) we will split an exponential complex number $e^{i\theta}$ into "half-angles", $e^{i\theta} = e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}}$; (b) we will think of 1 as $e^{i0}$ and split it into conjugates, $1 = e^{i0} = e^{i\frac{\theta}{2}}e^{-i\frac{\theta}{2}}$. Thereafter, we observe that $e^{i\frac{\theta}{2}}$ and $e^{i\frac{\theta}{2}}$ are conjugates. The formulas (1*) and (2*) on the previous page will enable further simplification.

$$\begin{aligned} e^{i\theta} + 1 &= e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}e^{-i\frac{\theta}{2}} \\ &= e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\right) \\ &= e^{i\frac{\theta}{2}}\left(2\cos\tfrac{\theta}{2}\right) \\ &= \left(2\cos\tfrac{\theta}{2}\right)e^{i\frac{\theta}{2}} \end{aligned}$$

$$\begin{aligned} e^{i\theta} - 1 &= e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}e^{-i\frac{\theta}{2}} \\ &= e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right) \\ &= e^{i\frac{\theta}{2}}\left(2i\sin\tfrac{\theta}{2}\right) \\ &= e^{i\frac{\theta}{2}}\left(2e^{i\frac{\pi}{2}}\sin\tfrac{\theta}{2}\right) \\ &= \left(2\sin\tfrac{\theta}{2}\right)e^{i\left(\frac{\pi}{2}+\frac{\theta}{2}\right)} \end{aligned}$$

*Concept check 4a: for the second example, why did we not leave the final answer as* $\left(2i\sin\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}$ *(as opposed to the first example with the final answer* $\left(2\cos\frac{\theta}{2}\right)e^{i\frac{\theta}{2}}$*)?*

*Concept check 4b: for the second example, how did we arrive at* $i = e^{i\frac{\pi}{2}}$*? (Hint: use an Argand diagram to visualize it)*

# Links and other resources

- Online version of these notes (with less explanation text) is available at math-atlas.vercel.app/notes/complex
- Computer generated questions: math-atlas.vercel.app/questions
- YouTube channel with worked TYS solutions and revision lectures http://tiny.cc/kelvinsoh
- Contact me at kelvinsohmath@gmail.com