# 13. Complex numbers

## Basics, complex conjugates

**Theory**

- A complex number $z$ is of the form $\boxed{z = x + yi}$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.
- We call $x$ the **real part** $\mathrm{Re}(z) = x$ and $y$ the **imaginary part** $\mathrm{Im}(z) = y$.
- The **complex conjugate** of $z = x + yi$ is given by $\boxed{z^* = x - yi}$.
  - $\boxed{z + z^* = 2x = 2\mathrm{Re}(z)}$
  - $\boxed{z - z^* = 2yi = 2i\mathrm{Im}(z)}$
  - $\boxed{zz^* = x^2 + y^2 = |z|^2}$

Example of complex division:

$$\frac{1-i}{3+4i} = \frac{1-i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{-1-7i}{25}$$

## The quadratic formula

**Example**

Solve $z^2 + 2z + 5 = 0$.

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

## Modulus/argument I

**Formula**

- $\boxed{r = |z| = \sqrt{x^2 + y^2}}$
- Let $\arg(z) = \theta$. $\boxed{\tan\theta = \frac{y}{x}}$

## Comparing parts

**Example**

Solve $z^2 + zz^* = 8 - 4i$.

Let $z = x + yi$

$(x + yi)^2 + (x + yi)(x - yi) = 8 - 4i$

$x^2 + 2xyi - y^2 + x^2 + y^2 = 8 - 4i$

$2x^2 + 2xyi = 8 - 4i$

Comparing real parts:

$2x^2 = 8 \;\Rightarrow\; x = \pm 2$

Comparing imaginary parts:

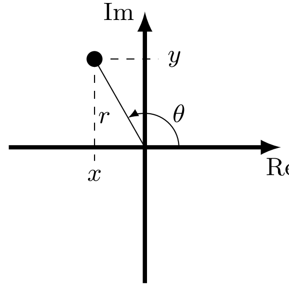
$2xy = -4 \;\Rightarrow\; y = \mp 1$

Hence $z = 2 - i$ or $z = -2 + i$.

## Modulus/argument II

Let $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$$\theta = \begin{cases} \alpha & \text{first quadrant} \\ \pi - \alpha & \text{second quadrant} \\ -(\pi - \alpha) & \text{third quadrant} \\ -\alpha & \text{fourth quadrant} \end{cases}$$




The Argand diagram

## Complex number forms

**Formula**

- **Cartesian form**: $\boxed{z = x + yi}$
- **Polar/trigo form**: $\boxed{z = r(\cos\theta + i\sin\theta)}$
- **Euler/exp form**: $\boxed{z = re^{i\theta}}$

## The conjugate root theorem

**Theory**

- Let $P(z)$ be a polynomial with **real coefficients**. The **conjugate root theorem** states that if $a + bi$ is a root to $P(z) = 0$, then its conjugate $a - bi$ is also a root to $P(z) = 0$.

**Example**

Solve $3z^3 - 7z^2 + 17z - 5 = 0$ given that $1 + 2i$ is a root.

Since all the coefficients are real, by the conjugate root theorem, $1 - 2i$ is also a root.

By the **factor theorem**, $(z - (1 + 2i))$ and $(z - (1 - 2i))$ are factors of the cubic polynomial.

$(z - 1 - 2i)(z - 1 + 2i) = (z - 1)^2 - (2i)^2 = z^2 - 2z + 5$.

By **long division** or **comparing coefficients**, we can obtain the final factor $3z - 1$.

$3z^3 - 7z^2 + 17z - 5 =$

$(z - (1 + 2i)(z - (1 - 2i))(3z - 1)$.

Hence $z = 1 + 2i, z = 1 - 2i$ or $z = \frac{1}{3}$.

## Modulus/argument III

$$|wz| = |w||z| \qquad \arg(wz) = \arg(w) + \arg(z)$$

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} \qquad \arg\left(\frac{w}{z}\right) = \arg(w) - \arg(z)$$

$$|z^n| = |z|^n \qquad \arg(z^n) = n\arg(z)$$

$$|z^*| = |z| \qquad \arg(z^*) = -\arg(z)$$

## The half-angle "trick"

$$\begin{aligned} 1 + e^{i2\theta} &= e^{i\theta}e^{-i\theta} + e^{i\theta}e^{i\theta} \\ &= e^{i\theta}\left(e^{-i\theta} + e^{i\theta}\right) \\ &= e^{i\theta}(2\mathrm{Re}(e^{i\theta})) \\ &= 2\cos\theta\, e^{i\theta} \end{aligned}$$

## Purely real/imaginary numbers

| Condition | Cartesian | Argument ($k \in \mathbb{Z}$) |
|---|---|---|
| real | $y = 0$ | $\arg = k\pi$ |
| real and positive | $y = 0,\ x > 0$ | $\arg = 2k\pi$ |
| real and negative | $y = 0,\ x < 0$ | $\arg = (2k+1)\pi$ |
| purely imaginary | $x = 0$ | $\arg = \frac{(2k+1)\pi}{2}$ |