13 Complex numbers

13.1 Cartesian form arithmetic

A complex number can be expressed in **cartesian form** $z = x + yi$. x is the **real part** of z, denoted $\mathrm{Re}(z)$, while y is the **imaginary part** of z, denoted $\mathrm{Im}(z)$.

Powers of i, addition, subtraction, multiplication

Since $i = \sqrt{-1}$, $\boxed{i^2 = -1}$. Then $i^3 = i^2, i = -i$ and $i^4 = (i^2)^2 = 1$.

$(2 - i) + (3 + 4i) = 5 + 3i$.

$(1 + 2i) - (3 - 5i) = -2 + 7i$.

$(2 + i)(3 + 4i) = 6 + 8i + 3i + 4i^2 = 6 + 11i - 4 = 2 + 11i$.

Complex conjugates, division

Given $z = x + yi$, its **complex conjugate**, z^*, is $\boxed{z^* = x - yi.}$

$$z + z^* = 2x = 2\mathrm{Re}(z) = 2\cos\theta \tag{1}$$
$$z - z^* = 2yi = 2\mathrm{Im}(z)i = 2i\sin\theta \tag{2}$$
$$\boxed{zz^* = x^2 + y^2 = |z|^2} \tag{3}$$

$$\frac{2+i}{3-4i} = \frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{2+11i}{3^2+4^2} = \frac{2+11i}{25} = \frac{2}{25} + \frac{11}{25}i.$$

13.2 Solving equations

Example 1. Solve the equation $(2 + 3i)z + 8 + 2i = 1 - i$.

Solution 1.

$$\begin{aligned}(2+3i)z &= 1 - i - (8 + 2i)\\ (2+3i)z &= -7 - 3i\\ &= \frac{-7-3i}{2+3i} \times \frac{2-3i}{2-3i}\\ &= \frac{-23+15i}{13}. \quad \blacksquare\end{aligned}$$

Quadratic equations

Example 2. Solve the equation $2z^2 - 2x + 5 = 0$.

Solution 2.

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{-36}}{4} \\ &= \frac{2 \pm 6i}{4} \\ &= \frac{1 \pm 3i}{2}. \quad \blacksquare \end{aligned}$$

Comparing real and imaginary parts

Remark: Comparison only works if all our variables are real.

Example 3. Solve the equation $-18iz + zz^* = -56 - 90i$.

Solution 3. Let $z = x + yi, \quad x, y \in \mathbb{R}$.

$$\begin{aligned} -18i(x + yi) + (x + yi)(x - yi) &= -56 - 90i \\ -18xi + 18y + x^2 + y^2 &= -56 - 90i \\ (y^2 + 18y + x^2) - 18xi &= -56 - 90i \end{aligned}$$

Comparing imaginary parts, $-18x = -90$ so $x = 5$.
Substituting $x = 5$ and comparing real parts,

$$\begin{aligned} y^2 + 18y + 25 &= -56 \\ y^2 + 18y + 81 &= 0 \\ (y + 9)^2 &= 0 \\ y &= -9 \end{aligned}$$

Hence $z = 5 - 9i$. $\blacksquare$

13.3 The conjugate root theorem

Theorem (Fundamental Theorem of Algebra)**.** *The polynomial equation* $az^n + bz^{n-1} + \ldots = 0, a \neq 0$ *has exactly* n *complex roots (including multiplicity).*

Theorem (Factor Theorem)**.** *If* $z = \alpha$ *is a root of a polynomial* $f(z)$ *(i.e.* $f(\alpha) = 0$*), then* $(z - \alpha)$ *is a factor of* $f(z)$*.*

Theorem (Conjugate Root Theorem)**.** *Let* $P(z)$ *be a polynomial with* ***real coefficients****. If* $z = a + bi$ *is a root of* $P(z) = 0$*, then its complex conjugate* $z^* = a - bi$ *is also a root.*

Expansion of factors from conjugate roots

$$\begin{aligned}(z - a + bi)(z - a - bi) &= (z - a)^2 - (bi)^2 \\ &= z^2 - 2a + a^2 + b^2\end{aligned}$$

$$\begin{aligned}(z - re^{i\theta})(z - re^{-i\theta}) &= z^2 - re^{i\theta}z - re^{-i\theta}z + re^{i\theta}re^{-i\theta} \\ &= z^2 - rz(re^{i\theta} + re^{-i\theta}) + r^2 \\ &= z^2 - (2r\cos\theta)z + r^2\end{aligned}$$

Solving polynomial equations cubic and above

Example 4. Consider the equation $2z^3 - z^2 + 14z + 30 = 0$.

(a) Verify that $1 + 3i$ is a root of the equation.

(b) Hence solve the equation.

Solution 4.
(a) We first evaluate $(1 + 3i)^2$ and $(1 + 3i)^3$.

$$\begin{aligned}(1 + 3i)^2 &= 1 + 6i + 9i^2 \\ &= -8 + 6i \\ (1 + 3i)^3 &= (1 + 3i)(-8 + 6i) \\ &= -8 + 18i^2 - 24i + 6i \\ &= -26 - 18i\end{aligned}$$

Substituting $z = 1 + 3i$ into the equation

$$\begin{aligned}\text{LHS} &= 2(-26 - 18i) - (-8 + 6i) + 14(1 + 3i) + 30\\ &= (-52 - 36i) + (8 - 6i) + (14 + 42i) + 30\\ &= 0\\ &= \text{RHS}\end{aligned}$$

Hence $1 + 3i$ is a root of the equation. ■

(b) Since the equation has only real coefficients, by the conjugate root theorem, $1 - 3i$ is also a root.

By the factor theorem, both $z - (1 + 3i) = (z - 1 - 3i)$ and $(z - 1 + 3i)$ are factors of $2z^3 - z^2 + 14z + 30$.

Expanding the two factors:

$$(z - 1 - 3i)(z - 1 + 3i) = z^2 - 2z + 10$$

We can then obtain the remaining factor either by **long division** or **comparing coefficients**.

$$\begin{array}{r|l} & 2z \quad +3 \\ \hline z^2 - 2z + 10 & 2z^3 - z^2 + 14z + 30 \\ & -2z^3 + 4z^2 - 20z \\ \hline & 3z^2 - 6z + 30 \\ & -3z^2 + 6z - 30 \\ \hline & 0 \end{array}$$

Let $2z^3 - z^2 + 14z + 30$
$= (z^2 - 2z + 10)(az + b)$
Comparing coefficients:

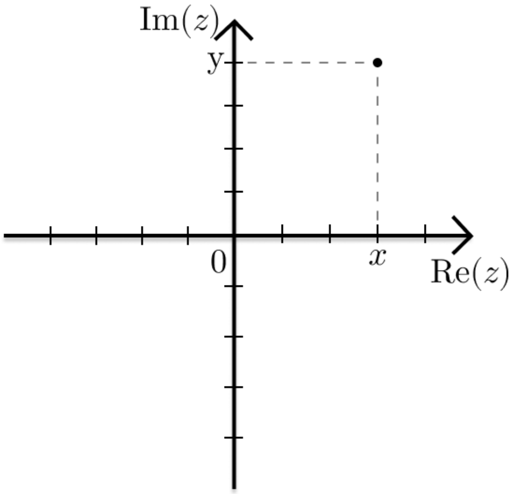
$$\begin{aligned} z^3 &: \quad a = 2 \\ z^0 &: \quad 10b = 30 \\ & \quad\;\; b = 3 \end{aligned}$$

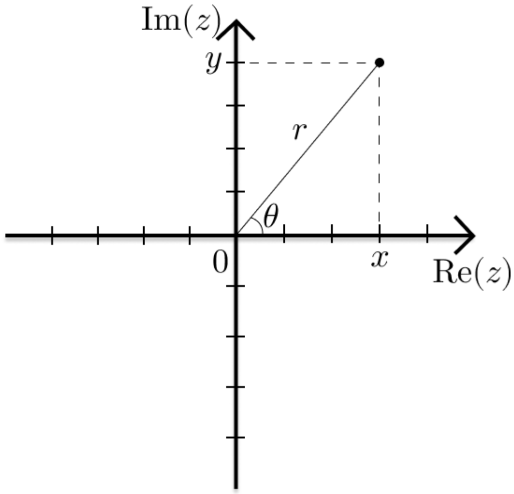
Hence the final factor is $(2z + 3)$.

$(z - 1 - 3i)(z - 1 + 3i)(2z + 3) = 0$.

$z = 1 + 3i,\ z = 1 - 3i$ or $z = -\frac{3}{2}$. ■

13.4 Argand diagram and polar form





The polar (modulus-argument) form

Three forms of a complex number

- Cartesian form $\boxed{z = x + yi.}$
- Trigonometric form $\boxed{z = r(\cos\theta + i\sin\theta).}$
- Exponential form $\boxed{z = re^{i\theta}.}$

Converting cartesian form to polar form

To calculate $|z| = r$ from x and y, $\boxed{|z| = r = \sqrt{x^2 + y^2}.}$

Meanwhile, $\arg(z) = \theta$ is related to x and y by $\tan\theta = \frac{y}{x}$.

$$\boxed{\alpha = \tan^{-1}\left|\frac{y}{x}\right|, \qquad \arg(z) = \theta = \begin{cases} \alpha & \text{if } x > 0, y > 0 \\ \pi - \alpha & \text{if } x < 0, y > 0 \\ -(\pi - \alpha) & \text{if } x < 0, y < 0 \\ -\alpha & \text{if } x > 0, y < 0 \end{cases}}$$

Example 5. (A) Convert $z = -1 + i$ into polar form.

Solution 5. $r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$.

$\alpha = \tan^{-1}|-1| = \frac{\pi}{4}$.

Since we are in the second quadrant, $\theta = \pi - \alpha = \frac{3\pi}{4}$.

Hence $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}e^{i\frac{3\pi}{4}}$. ■

Converting polar form to cartesian form

$\boxed{x = r\cos\theta,\ y = r\sin\theta.}$

Example 5. (B) Convert $z = 2e^{i\frac{2\pi}{3}}$ into polar form.

Solution 5.

$$\begin{aligned} 2e^{i\frac{2\pi}{3}} &= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \\ &= -1 + \sqrt{3}i. \quad \blacksquare \end{aligned}$$

13.5 Polar form arithmetic

$$\begin{aligned} wz &= \left(r_1e^{i\theta_1}\right)\cdot\left(r_1e^{i\theta_2}\right) = (r_1r_2)e^{i(\theta_1+\theta_2)} \\ \frac{w}{z} &= \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \\ z^n &= \left(re^{i\theta}\right)^n = r^ne^{in\theta} \\ z^* &= \left(re^{i\theta}\right)^* = re^{-i\theta} \end{aligned}$$

$$\begin{aligned} |wz| &= |w|\,|z| \\ \left|\frac{w}{z}\right| &= \frac{|w|}{|z|} \\ |z^n| &= |z|^n \\ |z^*| &= |z| \end{aligned}$$

$$\begin{aligned}\arg(wz) &= \arg(w) + (z) + 2k\pi,\\ \arg\left(\frac{w}{z}\right) &= \arg(w) - (z) + 2k\pi,\\ \arg(z^n) &= n\arg(z) + 2k\pi,\\ \arg(z^*) &= -\arg(z)\end{aligned}$$

where $k \in \mathbb{Z}$.

Example 6. Evaluate $\left(\dfrac{\left(e^{-\frac{\pi}{12}i}\right)^7 e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^*$.

Solution 6.

$$\begin{aligned}\left(\frac{\left(e^{-\frac{\pi}{12}i}\right)^7 e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^* &= \left(\frac{e^{-\frac{7\pi}{12}i}e^{-\frac{5\pi}{6}i}}{e^{\frac{\pi}{3}i}}\right)^*\\ &= \left(\frac{e^{-\frac{7\pi}{12}i+\left(-\frac{5\pi}{6}i\right)}}{e^{\frac{\pi}{3}i}}\right)^*\\ &= \left(\frac{e^{-\frac{17\pi}{12}i}}{e^{\frac{\pi}{3}i}}\right)^*\\ &= \left(e^{-\frac{17\pi}{12}i-\frac{\pi}{3}i}\right)^*\\ &= \left(e^{-\frac{7\pi}{4}i}\right)^*\\ &= e^{\frac{7\pi}{4}i}\\ &= e^{\left(\frac{7\pi}{4}-2\pi\right)i}\\ &= e^{-\frac{\pi}{4}i} \quad \blacksquare\end{aligned}$$

13.6 Real and purely imaginary numbers

Condition	Cartesian form, $x + yi$	Polar form, $re^{i\theta}$
Real	$y = 0$	$\theta = k\pi, \quad k \in \mathbb{Z}$
Real and positive	$y = 0, x > 0$	$\theta = 2k\pi, \quad k \in \mathbb{Z}$
Real and negative	$y = 0, x < 0$	$\theta = \pi + k\pi, \quad k \in \mathbb{Z}$
Purely imaginary	$x = 0$	$\theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

Example 7. Find the three smallest positive integers n such that $\left(e^{-\frac{\pi}{12}i}\right)^n$ is purely imaginary.

Solution 7.

$$\arg\left(\left(e^{-\frac{\pi}{12}i}\right)^n\right) = n\arg\left(e^{-\frac{\pi}{12}i}\right) = -\frac{n\pi}{12}.$$

For it to be purely imaginary,

$$\begin{aligned}-\frac{n\pi}{12} &= \frac{\pi}{2} + k\pi \\ -n\pi &= 6\pi + 12k\pi \\ n &= -6 - 12k\end{aligned}$$

The smallest positive integers n correspond to when $k = -1, -2, -3$.

Hence the three smallest positive integers $n = 6, 18, 24$. ■

13.7 Miscellaneous examples, techniques

Example 8. The complex number z is given by $z = 3 + bi$, where b is a real number.

(a) Find the possible values of b if $\dfrac{z^2}{z^*}$ is real.

For the rest of the question, it is further given that $b > 0$.

(b) Find the smallest integer value of n such that $|z^n| > 1000$.

(c) For the value of n found in (b), find the values of $|z^n|$ and $\arg(z^n)$ such that $-\pi < \arg(z^n) \le \pi$.

(d) On a single Argand diagram mark out the points A, B, D, D and E representing the complex numbers z, $\dfrac{z^2}{z}$, z^*, $\dfrac{18}{z}$ and $\dfrac{z^2}{6}$ respectively.

Solution 8.

(a) We first evaluate $\frac{z^2}{z^*}$ in terms of b.

$$\begin{aligned}\frac{z^2}{z^*} &= \frac{(3+bi)^2}{3-bi} \\ &= \frac{9-b^2+6bi}{3-bi}\cdot\frac{3+bi}{3+bi} \\ &= \frac{3(9-b^2)-6b^2+((9-b^2)b+18b)\,i}{9+b^2}\end{aligned}$$

Since $\frac{z^2}{z^*}$ is real, its imaginary part is 0.

$$\begin{aligned}(9-b^2)b+18b&=0\\ b\left(9-b^2+18\right)&=0\\ b(27-b^2)&=0\end{aligned}$$

Hence $b=0$ or $b=\pm\sqrt{27}=\pm 3\sqrt{3}$. ■

(b) Since $b>0, b=3\sqrt{3}$.

$$\begin{aligned}|z^n|&>1000\\ \left|3+3\sqrt{3}i\right|^n&>1000\\ \left(\sqrt{3^2+(3\sqrt{3})^2}\right)^n&>1000\\ 6^n&>1000\\ n\ln 6&>\ln 1000\\ n&>3.8553\end{aligned}$$

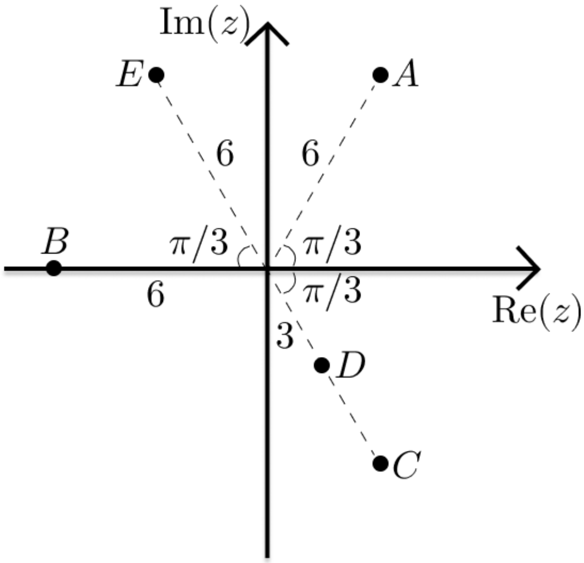
Hence the smallest integer value of $n=4$. ■

(c) For $z=3+3\sqrt{3}i$ and $n=4$,

$$\begin{aligned}|z^n|&=|z|^n\\ &=6^4\\ &=1296. \quad ■\end{aligned}$$

$$\begin{aligned}\arg(z)&=\tan^{-1}\left(\frac{3\sqrt{3}}{3}\right)\\ &=\frac{\pi}{3}\\ \arg(z^n)&=n\arg(z)\\ &\equiv 4\left(\frac{\pi}{3}\right)-2\pi\\ &=-\frac{2\pi}{3}. \quad ■\end{aligned}$$

(d) $z=6e^{i\frac{\pi}{3}}$, $\quad \dfrac{z^2}{z^*}=\dfrac{36e^{i\frac{2\pi}{3}}}{6e^{-i\frac{\pi}{3}}}=6e^{i\pi}$, $\quad z^*=6e^{-i\frac{\pi}{3}}$,
$\dfrac{18}{z}=\dfrac{18}{6e^{i\frac{\pi}{3}}}=3e^{-i\frac{\pi}{3}}$, $\quad \dfrac{z^2}{6}=\dfrac{36e^{i\frac{2\pi}{3}}}{6}=6e^{i\frac{2\pi}{3}}$.



The half-angle "trick"

$$
\begin{aligned}
e^{i\theta} + 1 &= e^{i\frac{\theta}{2}} e^{i\frac{\theta}{2}} + e^{i\frac{\theta}{2}} e^{-i\frac{\theta}{2}} \\
&= e^{i\frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} \right) \\
&= e^{i\frac{\theta}{2}} \left(2\cos\tfrac{\theta}{2} \right) \\
&= \left(2\cos\tfrac{\theta}{2} \right) e^{i\frac{\theta}{2}}
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} - 1 &= e^{i\frac{\theta}{2}} e^{i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} e^{-i\frac{\theta}{2}} \\
&= e^{i\frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right) \\
&= e^{i\frac{\theta}{2}} \left(2i\sin\tfrac{\theta}{2} \right) \\
&= e^{i\frac{\theta}{2}} \left(2e^{i\frac{\pi}{2}}\sin\tfrac{\theta}{2} \right) \\
&= \left(2\sin\tfrac{\theta}{2} \right) e^{i\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}
\end{aligned}
$$