21 Permutations and combinations

21.1 The multiplication principle

Our main objective in this chapter is to count: how many ways are there for an event to occur? Listing the possibilities out is one way we can do that, but this quickly gets out of hand for large numbers.

Our first general technique will then to break up our event into a sequence of steps, and apply the **multiplication principle.**

Technique 1 (the multiplication principle)**.**

Suppose we can break an event E up into two steps: step 1 **followed by** step 2. If there are m ways for step 1 to occur followed by n ways for step 2, then the total number of ways for E to occur is $\boxed{m \times n.}$

Example 1 (A: forming a number with repetition)**.**

How many ways are there to form a five-digit number satisfying the following conditions using only the numbers from 1,2,3,4,5,6,7. Repetitions are allowed.

(a) No restrictions.

(b) The number must be greater than 30,000.

(c) The number must be odd.

(d) The number must be greater than 30,000 and odd.

Solution 1.

(a) Step 1: there are 7 ways to fill in the first digit.
Step 2: there are 7 ways to fill in the second digit.

Repeating for all 5 digits,
required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5 = 16,807$.

(b) Step 1: there are 5 ways to fill the first digit (3,4,5,6,7).
Steps 2-5: there are 7 ways to fill in the subsequent digits.

Required number of ways $= 5 \times 7 \times 7 \times 7 \times 7 = 5 \times 7^4 = 12,005$.

(c) Step 1: there are 4 ways to fill the last digit (1,3,5,7).
Steps 2-5: there are 7 ways to fill in the subsequent digits.

Required number of ways $= 4 \times 7 \times 7 \times 7 \times 7 = 4 \times 7^4 = 9,604$.

(d) Step 1: there are 5 ways to fill the last digit (3,4,5,6,7).
Step 2: there are 4 ways to fill the last digit (1,3,5,7).
Steps 3-5: there are 7 ways to fill in the subsequent digits.

Required number of ways $= 5 \times 4 \times 7 \times 7 \times 7 = 5 \times 4 \times 7^3 = 6,860$. ■

21.2 Permutations: rearranging objects

We will often encounter situations where we will like to rearrange objects. For example, consider four objects A, B, C and D. To rearrange them, we can use the multiplication principle: there are 4 ways to decide which item will be first. Subsequently, there will be 3 ways to decide which item goes next, followed by 2 ways for the next, and 1 way for the last.

Hence the number of ways to rearrange 4 objects is given by $4 \times 3 \times 2 \times 1$. Since this technique occur so often, we give it a special name and symbol: the **factorial**, n!

Technique 2 (rearranging n objects using the factorial)**.**

There are $\boxed{n! \text{ ways}}$ to rearrange n unique objects in a line.

The factorial of a positive integer n, denoted $n!$, is defined by
$n! = n(n-1)(n-2)\cdots(1)$.

For example, $4! = 4(3)(2)(1) = 24$.

21.3 The addition principle

Use of the multiplication principle like in example 1A can be thought of as a "direct" method to count. However, this may not work for certain situations, especially when step 1 affects subsequent steps differently.

In such situations, we can consider using the **addition principle**, where we split our event up into disjoint **cases**.

> **Technique 3** (the addition principle)**.**
>
> Suppose we can break an event E up into two **cases**: case 1 **or** case 2 (with no overlap).
>
> If there are m ways for case 1 to occur and n ways for case 2, then the total number of ways for E to occur is $\boxed{m + n.}$

For example, in part (d) of the next question, we are asked to rearrange the digits $1, 2, 3, 4, 5$ (with no repetitions) to obtain a five-digit number that is greater than 20,000 and odd. One possible first step is to consider the first digit, which can be $2, 3, 4$ or 5 so that our number is greater than 20,000.

If we chose an even number to start with (2 or 4), when we fill in the last digit, we could go with $1, 3$ or 5. However, if we chose an odd number to start with (3 or 5), we only have two options left for the last digit (since either 3 or 5 is already used for the first).

The difference in step 2 for this example means that we cannot directly apply the multiplication principle. The addition principle is then ideal for such situations where we break our event up into cases.

Example 1 (B: forming a number by rearranging).

How many ways are there to rearrange the five digits $1, 2, 3, 4, 5$ to form a five-digit number satisfying the following conditions. Repetitions are **not** allowed.

(a) No restrictions.

(b) The number must be greater than 20,000.

(c) The number must be odd.

(d) The number must be greater than 20,000 and odd.

Solution 1.

(a) We rearrange the 5 digits to get $5! = 120$ ways.

(b) Step 1: there are 4 ways to fill the first digit (2,3,4,5).
Step 2: rearrange the remaining 4 digits.

Required number of ways $= 4 \times 4! = 96$.

(c) Step 1: there are 3 ways to fill the last digit (1,3,5).
Step 2: rearrange the remaining 4 digits.

Required number of ways $= 3 \times 4! = 72$.

(d) Case 1: Start with an odd number (3 or 5).
Number of ways $= 2 \times 2 \times 3! = 24$.
Case 2: Start with an even number (2 or 4).
Number of ways $= 2 \times 3 \times 3! = 36$.

Required number of ways $= 24 + 36 = 60$. ■

21.4 Combinations: choosing objects

$n!$ represents the number of ways to rearrange n distinct objects. That is, we will use all n objects. There are situations where we don't require all of them.

For example, consider three objects A, B and C. We may only require 2 of them. Listing the possibilities gives us $\{A, B\}, \{A, C\}$ or $\{B, C\}$ so there are 3 ways to choose 2 objects from 3 if the order is not important. If order is important (giving us additional cases $\{B, A\}, \{C, A\}, \{C, B\}$), then we have 6 ways.

Listing is way too cumbersome for larger possibilities so we introduce a new symbol and formula to calculate such situations. For choosing 2 objects out of 3, we have $\binom{3}{2} = 3$.

Technique 4 (choosing r objects out of n)**.**

There are $\boxed{\binom{n}{r} \text{ ways}}$ to choose r distinct objects out of n.
This is also denoted by ${}^nC_r$ and the objects are **unordered**.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

For an **ordered** selection, we have $\boxed{\binom{n}{r} \times r! \text{ ways.}}$
This is also denoted by ${}^nP_r$.

Example 1 (C: forming a number by choosing)**.**

How many ways are there to use the digits from $1, 2, 3, 4, 5, 6, 7$ to form a five-digit number satisfying the following conditions. Repetitions are **not** allowed.

(a) No restrictions.

(b) The number must be greater than 30,000.

(c) The number must be odd.

(d) The number must be greater than 30,000 and odd.

Solution 1.

(a) Number of ways $= \binom{7}{5} \times 5! = 2520$.

(b) Number of ways $= 5 \times \binom{6}{4} \times 4! = 1800$.

(c) Number of ways $= 4 \times \binom{6}{4} \times 4! = 1440$.

(d) Case 1: Start with an odd number (3, 5 or 7).
Number of ways $= 3 \times 3 \times \binom{5}{3} \times 3! = 540$.
Case 2: Start with an even number (4 or 6).
Number of ways $= 2 \times 4 \times \binom{5}{3} \times 3! = 480$.

Required number of ways $= 540 + 480 = 1020$. ■

21.5 Complements

Sometimes, the "opposite" of an event could be easier to count. In such cases, we can use the technique of complements. Consider an event E. The **complement** of the event is denoted by E' and represents the set of outcomes **not** in E.

Technique 5 (complements)**.**

Let $n(E)$ represent the number of ways for an event E to occur and $n(\text{total})$ represent the total number of ways for everything to occur with no restrictions. If $n(E')$ represents the number of ways for E **not** to occur, then $\boxed{n(E) = n(\text{total}) - n(E').}$

Example 2 (forming groups)**.**

A group of students is to be chosen to represent three schools, A, B and C. The group is to consist of 10 students, and is chosen from a set of 15 students consisting of 2 from A, 4 from B, and 9 from C.

Find the number of ways in which the group can be chosen if it includes

(a) 1 student from A, 3 from B and 6 from C,

(b) students from B and C only,.

(c) at least 8 students from C,

(d) at least 1 student from each school.

Solution 2.

(a) Required number of ways $= \binom{2}{1} \times \binom{4}{3} \times \binom{9}{6} = 672.$

(b) We only consider the $4 + 9 = 13$ students from B and C.
Required number of ways $= \binom{13}{10} = 286.$

(c) Case 1: exactly 8 students from C, 2 from A and B.
Case 2: exactly 9 students from C, 1 from A and B.
Required number of ways $= \binom{9}{8} \times \binom{6}{2} + \binom{9}{9} \times \binom{6}{1} = 135 + 6 = 141.$

(d) We consider the complement, which involves *some* school not being represented.

Complement case 1: A not represented. Number of ways $= \binom{13}{10} = 286.$

Complement case 2: B not represented. Number of ways $= \binom{11}{10} = 11.$

Total number of ways with no restrictions $= \binom{15}{10}.$

Required number of ways $= \binom{15}{10} - (286 + 11) = 2706.$ ■

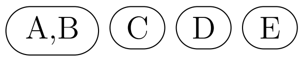
21.6 Grouping, slotting and alternating

Grouping

For situations where we want objects to be **together**, we can employ the **grouping** method.

Technique 6 (the grouping method)**.**

Let us consider the example of rearranging five objects A,B,C,D and E, where A and B must be together.



Step 1: Consider A and B grouped together as if they are just one object. Then we have 4! ways to rearrange the four overall objects.
Step 2: Rearrange A and B among themselves: 2! ways.

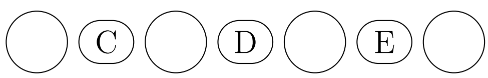
Hence the number of ways $= 4! \times 2!$.

Slotting

For situations where we want objects to be **separated**, we can employ the **slotting** method.

Technique 7 (the slotting method)**.**

Let us consider the example of rearranging five objects A,B,C,D and E, where A and B must be separated.



Step 1: Rearrange the remaining 3 objects (C,D,E) first: 3! ways.
Step 2: We observe that there are 4 "slots" that A and B can then go into if they are to be separated. Choosing 2 slots for them and rearranging among themselves, we have $\binom{4}{2} \times 2!$ ways.

Hence the number of ways $= 3! \times \binom{4}{2} \times 2!$.

Slotting vs complement

In addition to the slotting method, the complement method can also be used. Take the earlier example of rearranging five objects A,B,C,D and E, where A and B must be separated. Using the slotting method gives us $3! \times \binom{4}{2} \times 2! = 72$ ways.

If we consider the complement of A and B separated, that means A and B must be together. Then we can use the complement and grouping methods to get $5! - 4! \times 2! = 120 - 48 = 72$ ways.

Unsurprising, both methods give us the same answer: they are simply two ways of counting the same event.

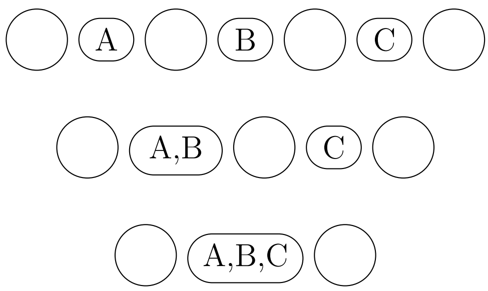
However, now let us consider a case of three objects A,B,C out of six A,B,C,D,E,F.

The slotting method gives us $3! \times \binom{4}{3} \times 3! = 144$ ways.

The complement method gives us $6! - (4! \times 3!) = 576$ ways.

They are different because the slotting method gives us a situation where A,B,C are **separated/not next to one another**. This is illustrated in the first diagram below.

Meanwhile, the complement method gives us a situation where A,B,C are **not all together**. We cannot have all three of them together (the third diagram below) but two of them together is allowed as well (the second diagram below). This is in contrast to the slotting method where only the first case is allowed (and is thus a subset, giving us a smaller number).



Alternating

We can modify the slotting approach for cases where we want object to **alternate**. Unlike the slotting method, where there is a choosing step to decide on the slots, alternating is a lot more restrictive. Depending on the number of objects, and whether we are arranging in a line or a circle, the number of ways to alternate is either 1 or 2.

Technique 8 (alternating arrangements)**.**

Let us consider the example of three men A,B,C and three women D,E,F and we will like to rearrange the six people in a straight line where men and women alternate.

Step 1: Rearrange the 3 women first: 3! ways.
Step 2: We observe that there are we can either have a man first or a woman first to alternate: MFMFMF or FMFMFM. Thus there is 2 ways to alternate.
Step 3: Rearrange the 3 men among themselves: 3! ways.

Hence the number of ways to alternate the men and women $= 3! \times 2 \times 3!$.

Example 3 (A: rearranging people (in a line))**.**

A group of 6 people consists of 3 married couples. Find the number of different possible orders for the group to stand in a line if

(a) there are no restrictions,

(b) each married man stands next to his wife,

(c) the men and women alternate,

(d) each married man stands next to his wife and the men and women alternate.

Solution 3.

(a) Required number of ways $6! = 720$.

(b) We group each couple, so we have 3 objects overall. We can also rearrange the couple among themselves for each couple.

Required number of ways $3! \times 2! \times 2! \times 2! = 48$.

(c) We rearrange the men and women separately, and observe that there are 2 ways to alternate (either men first or women first).

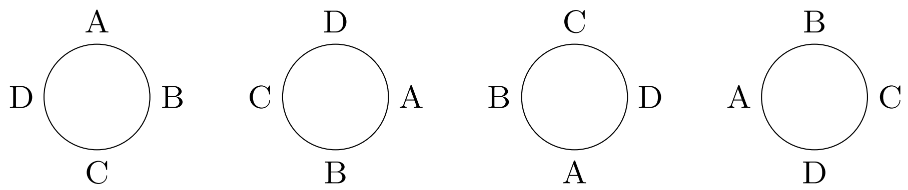
Required number of ways $= 3! \times 2 \times 3! = 72$.

(d) We group each couple and consider 3 overall objects. Since men and women must alternate, we can only rearrange the first couple (man or woman first). Subsequent couples cannot be rearranged because they must follow the arrangement from the first couple for men and women to alternate.

Required number of ways $3! \times 2 = 12$. ■

21.7 Rearranging objects in a circle

As opposed to a straight line, there are less ways to rearrange objects in a circle because the circle is identical under rotation (refer to figure below with four identical arrangements).



One way to count arrangements in a circle is to arbitrarily select one object as the "start" of our circle. We are thus left to rearrange the remaining $n-1$ objects as if they are on a straight line.

Technique 9 (circular arrangements)**.**

There are $\boxed{(n-1)! \text{ ways}}$ to rearrange n unique objects in a circle.

Example 3 (B: rearranging people (in a circle))**.**

A group of 6 people consists of 3 married couples. Find the number of different possible orders for the group to stand in a circle if

(a) there are no restrictions,

(b) each married man stands next to his wife,

(c) the men and women alternate,

(d) each married man stands next to his wife and the men and women alternate.

Solution 3.

(a) Required number of ways $(6-1)! = 120$.

(b) We group each couple, so we have 3 objects overall. We can also rearrange the couple among themselves for each couple.

Required number of ways $(3-1)! \times 2! \times 2! \times 2! = 16$.

(c) We rearrange the men and women separately, and observe that there is only 1 way to alternate in a circle.

Required number of ways $= (3-1)! \times 1 \times 3! = 12$.

(d) We group each couple and consider 3 overall objects. Since men and women must alternate, we can only rearrange the first couple (man or woman first). Subsequent couples cannot be rearranged because they must follow the arrangement from the first couple for men and women to alternate.

Required number of ways $(3-1)! \times 2 = 4$. ■

21.8 Rearranging identical objects

So far, all our discussion has been about unique/distinct objects. When objects are identical, there are lesser ways to rearrange them (for example, the arrangements A_1A_2B and A_2A_1B are considered the same if A_1 is the same as A_2.

Technique 10 (rearranging identical objects).

There are $\boxed{\dfrac{n!}{r_1!r_2!\cdots r_k!} \text{ ways}}$ to rearrange n objects in a line, where there are r_1 **identical** objects of type 1, r_2 identical objects of type 2, etc. For a circle, there are $\boxed{\dfrac{(n-1)!}{r_1!r_2!\cdots r_k!} \text{ ways.}}$

e.g. there are $\dfrac{9!}{3!2!2!}$ ways to rearrange AAABCCDEE in a line.

Example 4 (A: rearranging identical objects).

Find the number of ways in which the letters of the word SEQUENCE can be arranged if

(a) there are no restrictions,

(b) S and Q must not be next to one another,

(c) between any two Es there must be at least 2 other letters.

Solution 4.

(a) Since P is repeated three times, required number of ways $= \frac{8!}{3!} = 6720$.

(b) Using the complement, required number of ways $= 6720 - \frac{7!\times 2!}{3!} = 5040$.

(c) Case 1: □E□□E□□E
Case 2: E□□E□□E□
Case 3: E□□□E□□E
Case 4: E□□E□□□E

Number of ways for each case = 5! (rearrange the remaining five letters)
Required number of ways $= 5! \times 4 = 480$. ■

21.9 Choosing identical objects

Unlike rearranging identical objects in the previous section, there is no simple formula that can handle combinations of identical objects. This is because when we are not utilizing all objects in a bigger set, sometimes identical objects are chosen, but sometimes they are not (for example, we can choose AAB or ABC from AABCD).

Hence, cases is often employed to tackle situations where we need to choose identical objects, as seen in the following example.

Example 4 (B: choosing identical objects)**.**

A "codeword" is to be formed using the letters of COMPOSITION. Find the number of possible "codeword"s that are made up of

(a) 3 letters,

(b) 4 letters.

Solution 4.

(a) Case 1: no repeats ($\alpha\beta\gamma$): $\binom{8}{3} \times 3! = 336$

Case 2: repeats twice ($\alpha\alpha\beta$): $\binom{2}{1} \times \binom{7}{1} \times \frac{3!}{2!} = 42$

Case 3: repeats thrice ($\alpha\alpha\alpha$): 1

Required number of ways $336 + 42 + 1 = 379$.

(b) Case 1: no repeats ($\alpha\beta\gamma\delta$): $\binom{8}{4} \times 4! = 1680$

Case 2: one letter repeats twice ($\alpha\alpha\beta\gamma$): $\binom{2}{1} \times \binom{7}{2} \times \frac{4!}{2!} = 504$

Case 3: two letters repeats twice ($\alpha\alpha\beta\beta$): $\binom{2}{2} \times \frac{4!}{2!2!} = 6$

Case 4: repeats thrice ($\alpha\alpha\alpha\beta$): $1 \times \binom{7}{1} \times \frac{4!}{3!} = 28$

Required number of ways $1680 + 504 + 6 + 28 = 2218$. ■

Links and other resources

- Online version of these notes (with less explanation text) is available at math-atlas.vercel.app/notes/pnc
- Computer generated questions: math-atlas.vercel.app/questions
- YouTube channel with worked TYS solutions and revision lectures http://tiny.cc/kelvinsoh
- Contact me at kelvinsohmath@gmail.com