21. Permutations and combinations

Summary of techniques

Techniques

- **The multiplication principle**:
  By breaking an event up into **steps** with m ways for step 1, followed by n ways for step 2, there are $\boxed{m \times n}$ ways in total.
- **The addition principle**:
  By breaking an event up into **cases** with either case 1 (m ways) or case 2 (n ways), there are $\boxed{m + n}$ ways in total.
- **Complements**:
  $\boxed{n(A) = n(\text{total}) - n(A')}$
- **Permutation (line)**:
  There are $\boxed{n!}$ ways to **rearrange** n unique objects in a **line**.
- **Permutation (circle)**:
  There are $\boxed{(n-1)!}$ ways for a **circle**.
- **Permutation (identical objects)**:
  There are $\boxed{\dfrac{n!}{r_1! r_2! \cdots r_k!}}$ ways to rearrange n objects in a line, of which r_1 objects of type 1 are **identical**, r_2 of type 2 are identical, etc. For a circle, there are $\boxed{\dfrac{(n-1)!}{r_1! r_2! \cdots r_k!}}$ ways.
- **Combination**:
  There are $\boxed{\dbinom{n}{r}}$ ways to **choose** r unique objects out of n, where the objects are **unordered**.
  For an **ordered** selection, there are $\boxed{\dbinom{n}{r} \times r!}$ ways.
- **Grouping**: (A,B) (C) (D) (E) $5! \times 2!$
- **Slotting**: ( ) (C) ( ) (D) ( ) (E) ( ) $3! \times \binom{4}{2} \times 2!$
- **Alternating**: MFMFMF or FMFMFM. $3! \times 2 \times 3!$

Example 1: forming a number

Example

How many ways are there to form a five-digit number with
(a) no restrictions, (b) the number being greater than 30,000 and odd.

Case A: use the numbers from $1, 2, 3, 4, 5, 6, 7$. Repetitions are allowed.
Case B: use the numbers from $1, 2, 3, 4, 5$. Repetitions are not allowed.
Case C: use the numbers from $1, 2, 3, 4, 5, 6, 7$. Repetitions are not allowed.

Case A:
(a) 7^5
(b) $5 \times 4 \times 7^3$

Case B:
(a) $5!$
(b) Case 1: start with odd number,
Case 2: start with even number.
$2 \times 2 \times 3! + 1 \times 3 \times 3!$

Case C:
(a) $\binom{7}{5} \times 5!$
(b) Case 1: start with odd number,
Case 2: start with even number.
$3 \times 3 \times \binom{5}{3} \times 3! + 2 \times 4 \times \binom{5}{3} \times 3!$

Example 2: forming groups

Example

A group of 10 students is to be chosen from a set of 15 consisting of 2 from school A, 4 from B and 9 from C.

Find the number of ways in which the group can be chosen if it includes
(a) 1 from A, 3 from B and 6 from C,
(b) at least 1 student from each school.

(a) $\binom{2}{1} \times \binom{4}{3} \times \binom{9}{6}$

(b) We consider the complement, which involves some school not being represented. There are two cases: A not represented or B not represented.
$\binom{15}{10} - \left(\binom{13}{10} + \binom{11}{10}\right)$

Example 3: rearranging people

Example

A group of 6 people consists of 3 married couples. Find the number of possible orders for the group to stand

(a) in a circle,
(b) in a line such that each married man stands next to his wife,
(c) in a line such that men and women alternate.

(a) $5!$

(b) $3! \times 2! \times 2! \times 2!$

(b) $3! \times 2 \times 3!$

Example 4: identical objects

(a) There are $\frac{11!}{3!2!}$ ways to rearrange the letters of the word COMPOSITION.

(b) To pick a 3 letter codeword,
Case 1: no repeats: $\binom{8}{3} \times 3!$,
Case 2: repeat twice: $\binom{2}{1} \times \binom{7}{1} \times \frac{3!}{2!}$,
Case 3: repeat thrice: 1.