# 21 Permutations and combinations

## Summary of techniques

1. **The multiplication principle**:
   By breaking an event up into **steps** with $m$ ways for step 1, followed by $n$ ways for step 2, there are $\boxed{m \times n}$ ways for the event to occur.

2. **The addition principle**:
   By breaking an event up into **cases** with either case 1 ($m$ ways) or case 2 ($n$ ways), there are $\boxed{m + n}$ ways for the event to occur.

3. **Complements**:
   $\boxed{n(A) = n(\text{total}) - n(A')}$

4. **Permutation (line)**:
   There are $\boxed{n!}$ ways to **rearrange** $n$ unique objects in a **line**.

5. **Permutation (circle)**:
   There are $\boxed{(n-1)!}$ ways to rearrange $n$ unique objects in a **circle**.

6. **Permutation (identical objects)**:
   There are $\boxed{\dfrac{n!}{r_1!r_2!\cdots r_k!}}$ ways to rearrange $n$ objects in a line, of which $r_1$ objects of type 1 are **identical**, $r_2$ of type 2 are identical, etc.
   For a circle, there are $\boxed{\dfrac{(n-1)!}{r_1!r_2!\cdots r_k!}}$ ways.

7. **Combination**:
   There are $\boxed{\dbinom{n}{r}}$ ways to **choose** $r$ unique objects out of $n$, where the objects are **unordered**.
   For an **ordered** selection, there are $\boxed{\dbinom{n}{r} \times r!}$ ways.

8. **Grouping**: (A,B) (C) (D) (E)    $5! \times 2!$

9. **Slotting**: ( ) (C) ( ) (D) ( ) (E) ( )    $3! \times \binom{4}{2} \times 2!$

10. **Alternating**: MFMFMF or FMFMFM.    $3! \times 2 \times 3!$

## 21.1 The multiplication principle

**Example 1** (A: forming a number with repetition).

How many ways are there to form a five-digit number satisfying the following conditions using only the numbers from 1,2,3,4,5,6,7. Repetitions are allowed.

(a) No restrictions.

(b) The number must be greater than 30,000.

(c) The number must be odd.

(d) The number must be greater than 30,000 and odd.

**Solution 1.**

(a) Required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5 = 16,807$.

(b) Required number of ways $= 5 \times 7 \times 7 \times 7 \times 7 = 5 \times 7^4 = 12,005$.

(c) Required number of ways $= 4 \times 7 \times 7 \times 7 \times 7 = 4 \times 7^4 = 9,604$.

(d) Required number of ways $= 5 \times 4 \times 7 \times 7 \times 7 = 5 \times 4 \times 7^3 = 6,860$. ■

## 21.2 Permutations: rearranging objects

**Example 1** (B: forming a number by rearranging).

How many ways are there to rearrange the five digits $1, 2, 3, 4, 5$ to form a five-digit number satisfying the following conditions. Repetitions are **not** allowed.

(a) No restrictions.

(b) The number must be greater than 20,000.

(c) The number must be odd.

(d) The number must be greater than 20,000 and odd.

**Solution 1.**

(a) We rearrange the 5 digits to get $5! = 120$ ways.

(b) Required number of ways $= 4 \times 4! = 96$.

(c) Required number of ways $= 3 \times 4! = 72$.

(d) Case 1: Start with an odd number (3 or 5).
Number of ways $= 2 \times 2 \times 3! = 24$.
Case 2: Start with an even number (2 or 4).
Number of ways $= 2 \times 3 \times 3! = 36$.

Required number of ways $= 24 + 36 = 60$. ■

## 21.4 Combinations: choosing objects

**Example 1** (C: forming a number by choosing).

How many ways are there to use the digits from $1, 2, 3, 4, 5, 6, 7$ to form a five-digit number satisfying the following conditions. Repetitions are **not** allowed.

(a) No restrictions.

(b) The number must be greater than 30,000.

(c) The number must be odd.

(d) The number must be greater than 30,000 and odd.

**Solution 1.**

(a) Number of ways $= \binom{7}{5} \times 5! = 2520$.

(b) Number of ways $= 5 \times \binom{6}{4} \times 4! = 1800$.

(c) Number of ways $= 4 \times \binom{6}{4} \times 4! = 1440$.

(d) Case 1: Start with an odd number (3, 5 or 7).
Number of ways $= 3 \times 3 \times \binom{5}{3} \times 3! = 540$.
Case 2: Start with an even number (4 or 6).
Number of ways $= 2 \times 4 \times \binom{5}{3} \times 3! = 480$.

Required number of ways $= 540 + 480 = 1020$. ■

## 21.5 Complements

**Example 2** (forming groups).

A group of students is to be chosen to represent three schools, $A, B$ and $C$. The group is to consist of 10 students, and is chosen from a set of 15 students consisting of 2 from $A$, 4 from $B$, and 9 from $C$.

Find the number of ways in which the group can be chosen if it includes

(a) 1 student from $A$, 3 from $B$ and 6 from $C$,

(b) students from $B$ and $C$ only,.

(c) at least 8 students from $C$,

(d) at least 1 student from each school.

**Solution 2.**

(a) Required number of ways $= \binom{2}{1} \times \binom{4}{3} \times \binom{9}{6} = 672.$

(b) We only consider the $4 + 9 = 13$ students from $B$ and $C$.
Required number of ways $= \binom{13}{10} = 286.$

(c) Case 1: exactly 8 students from $C$, 2 from $A$ and $B$.
Case 2: exactly 9 students from $C$, 1 from $A$ and $B$.
Required number of ways $= \binom{9}{8} \times \binom{6}{2} + \binom{9}{9} \times \binom{6}{1} = 135 + 6 = 141.$

(d) We consider the complement, which involves *some* school not being represented.
Complement case 1: $A$ not represented. Number of ways $= \binom{13}{10} = 286.$
Complement case 2: $B$ not represented. Number of ways $= \binom{11}{10} = 11.$
Total number of ways with no restrictions $= \binom{15}{10}$.
Required number of ways $= \binom{15}{10} - (286 + 11) = 2706.$ ■

# 21.6 Grouping, slotting and alternating

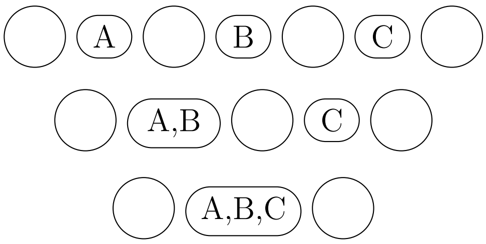
## Slotting vs complement

Let us consider a case of three objects A,B,C out of six A,B,C,D,E,F.

To have A,B,C **separated/not next to one another**, we use the slotting method to get $3! \times \binom{4}{3} \times 3! = 144$ ways, which is illustrated by the first figure below.

To have A,B,C **all together**, we use the grouping method to get $4! \times 3!$ ways. This is illustrated by the third figure below.

To have A,B,C is **not all together** we use the complement method to get $6! - (4! \times 3!) = 576$ ways. This is illustrated by the first and second figures.




**Example 3** (A: rearranging people (in a line))**.**

A group of 6 people consists of 3 married couples. Find the number of different possible orders for the group to stand in a line if

(a) there are no restrictions,

(b) each married man stands next to his wife,

(c) the men and women alternate,

(d) each married man stands next to his wife and the men and women alternate.

**Solution 3.**

(a) Required number of ways $6! = 720$.

(b) Required number of ways $3! \times 2! \times 2! \times 2! = 48$.

(c) Required number of ways $= 3! \times 2 \times 3! = 72$.

(d) Required number of ways $3! \times 2 = 12$. ■

## 21.7 Rearranging objects in a circle

**Example 3** (B: rearranging people (in a circle)).

A group of 6 people consists of 3 married couples. Find the number of different possible orders for the group to stand in a circle if

(a) there are no restrictions,

(b) each married man stands next to his wife,

(c) the men and women alternate,

(d) each married man stands next to his wife and the men and women alternate.

**Solution 3.**

(a) Required number of ways $(6-1)! = 120$.

(b) Required number of ways $(3-1)! \times 2! \times 2! \times 2! = 16$.

(c) Required number of ways $= (3-1)! \times 1 \times 3! = 12$.

(d) Required number of ways $(3-1)! \times 2 = 4$. ■

## 21.8 Rearranging identical objects

**Example 4** (A: rearranging identical objects)**.**

Find the number of ways in which the letters of the word SEQUENCE can be arranged if

(a) there are no restrictions,

(b) S and Q must not be next to one another,

(c) between any two Es there must be at least 2 other letters.

**Solution 4.**

(a) Since P is repeated three times, required number of ways $= \frac{8!}{3!} = 6720$.

(b) Using the complement, required number of ways $= 6720 - \frac{7! \times 2!}{3!} = 5040$.

(c) Case 1: □E□□E□□E
Case 2: E□□E□□E□
Case 3: E□□□E□□E
Case 4: E□□E□□□E

Number of ways for each case $= 5!$ (rearrange the remaining five letters)
Required number of ways $= 5! \times 4 = 480$. ■

## 21.9 Choosing identical objects

**Example 4** (B: choosing identical objects)**.**

A "codeword" is to be formed using the letters of COMPOSITION. Find the number of possible "codeword"s that are made up of

(a) 3 letters,

(b) 4 letters.

**Solution 4.**

(a) Case 1: no repeats ($\alpha\beta\gamma$): $\binom{8}{3} \times 3! = 336$

Case 2: repeats twice ($\alpha\alpha\beta$): $\binom{2}{1} \times \binom{7}{1} \times \frac{3!}{2!} = 42$

Case 3: repeats thrice ($\alpha\alpha\alpha$): 1

Required number of ways $336 + 42 + 1 = 379$.

(b) Case 1: no repeats ($\alpha\beta\gamma\delta$): $\binom{8}{4} \times 4! = 1680$

Case 2: one letter repeats twice ($\alpha\alpha\beta\gamma$): $\binom{2}{1} \times \binom{7}{2} \times \frac{4!}{2!} = 504$

Case 3: two letters repeats twice ($\alpha\alpha\beta\beta$): $\binom{2}{2} \times \frac{4!}{2!2!} = 6$

Case 4: repeats thrice ($\alpha\alpha\alpha\beta$): $1 \times \binom{7}{1} \times \frac{4!}{3!} = 28$

Required number of ways $1680 + 504 + 6 + 28 = 2218$. ■