22 Probability

22.1 Basics and notation

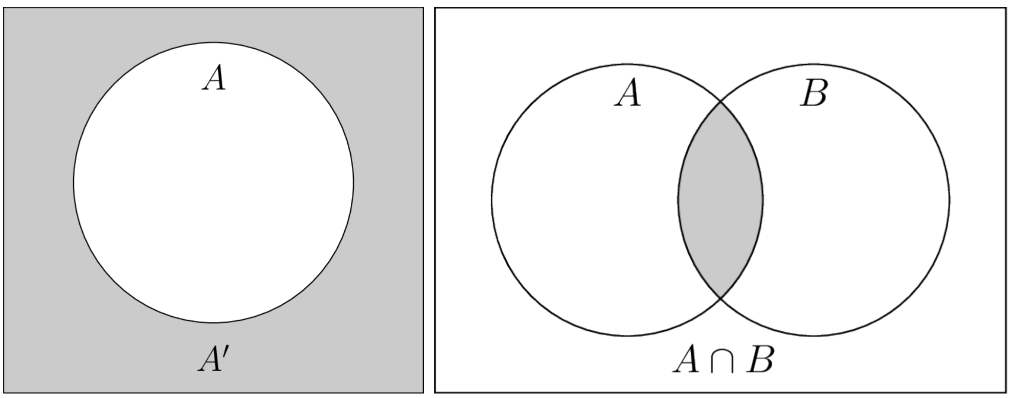
In this chapter we investigate the **probability**, or chance, of events. We use $P(A)$ to denote the probability that event A occurs. For example, $P(\text{heads}) = \frac{1}{2}$ when flipping an unbiased coin.

$0 \leq P(A) \leq 1$ for all events A. $P(A) = 0$ if there are no outcomes that are possible for A. For example, $P(\text{die roll shows 7}) = 0$ when rolling a regular six-sided die. $P(A) = 1$ if A contains all possible outcomes. For example, $P(\text{die roll shows less than 7}) = 1$ when rolling a regular six-sided die.

Set notation

Set notation can be a convenient way to talk about combination of events. As an illustrative example for this subsection, let us consider rolling a fair six-sided die. Let A denote the event that the die roll shows a number that is at least 5, B denote the event that the die roll shows an odd number, and C denote the event that the die roll shows a number that is 6.

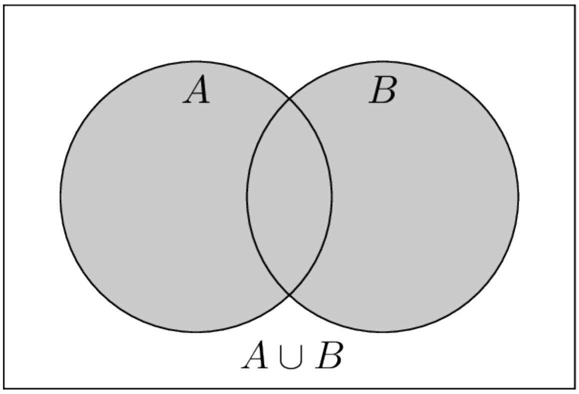
The **complement**, A' denotes the event that A does **not** occur. For our die roll example, A' will denote the event that the die roll shows a number that is less than 5. Since all probabilities add up to 1, we have a useful formula $\boxed{P(A') = 1 - P(A)}$.

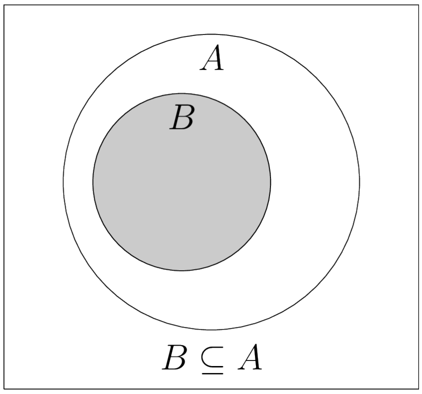


The **intersection**, $A \cap B$ denotes the event that A **and** B occur. For our die roll example, $A \cap B$ will denote the event that the die roll shows 5.

The **union**, $A \cup B$ denotes the event that A **or** B occur (or both). For our die roll example, $A \cup B$ will denote the event that the die roll shows either 1, 3, 5 or 6. A useful formula for the union:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$





Sometimes we may be interested in **inequalities** for our probabilities. The concept of **subsets** can be useful. $\boxed{D_1 \subseteq D_2 \implies P(D_1) \leq P(D_2)}$. For our die roll example, notice that C is a subset of A so $P(C) \leq P(A)$.

Adding and multiplying probabilities

Similar to the addition principle in the chapter of permutations and combinations, we **add** probabilities if we can break an event up into **cases** that **do not overlap**. For our die roll example where A denote the event that the die roll shows a number that is at least 5, we can break A up into either the die roll showing 5 or the die roll showing 6. Each case happens with probability $\frac{1}{6}$. Adding them up gives us $P(A) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

When events overlap, however, adding can lead to "over-counting". The union formula as well as tools like tree and Venn diagrams can help us navigate these more complicated situations.

Meanwhile, similar to the multiplication principle in P&C, we **multiply** probabilities if we break an event up into **steps**. Let us now consider rolling a fair six-sided die twice and let A be the event that we get a sum of 12 from the two rolls. We can break A up into two steps: getting a 6 on the first roll followed another 6 on the second roll. We multiply the probabilities to get $P(A) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Multiplying probabilities can get tricky if the first step "affects" the probability of the second step. This is where the idea of conditional probability in the next section comes in.

22.2 Conditional probability

Probabilities "change" depending on the information we have.

For example, consider a box with 4 red balls and 2 green balls from which we draw without replacement. Let A be the event that we draw a green ball on the second draw and B be the event we draw a red ball on the first draw. $P(B) = \frac{4}{6} = \frac{2}{3}$.

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If we knew that a red ball was drawn first, then there will be 3 red balls and 2 green balls left for the second draw. The probability of ending up with a green ball on the second draw, given that a red ball was drawn first, is given by $P(A \mid B) = \frac{2}{5}$.

In a similar fashion, if we knew that a green ball was drawn first instead, then the probability of ending up with a green ball on the second draw is $P(A \mid B') = \frac{1}{5}$.

Meanwhile, the probability of drawing a red ball on the first draw on the first draw and a green ball on the second draw is $P(A \cap B) = \frac{4}{6} \times \frac{2}{5} = \frac{4}{15}$.

Generalization of the example above gives us a very useful **conditional probability** formula $\boxed{P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}}$.

Concept check: are you able to explain the difference between what $P(A \mid B)$ and $P(B \mid A)$ mean? Are you able to calculate the latter? What about $P(A)$?

22.3 Tree diagram

A **tree diagram** is a very useful tool to help us visualize many problem situations. It is especially useful to help us keep track of conditional probabilities as well as remind us when to add or multiply probabilities.

Some useful tips when using a tree diagram (refer to the example below to see it in action):

- Each time we branch out, the sum of probability of the individual branches should be 1.
- We calculate probabilities by starting from the 'root' of the tree and travel 'down'. Probabilities at each step are multiplied to obtain the overall probability of the 'route'.
- If we want to combine multiple routes down the tree, we will add the probabilities of each 'route' (each 'route' correspond to an individual case in our analysis).

Example 1 (tree diagram).

There is a probability of 0.6 that Daniel takes his parents' car to go to school. Otherwise, he will take a bus.

If he takes his parents' car, he has a 0.01 probability of being late. The conditional probability that Daniel is late for school given that he took a bus is 0.05.

Find the probability that

(a) Daniel arrives to school on time,

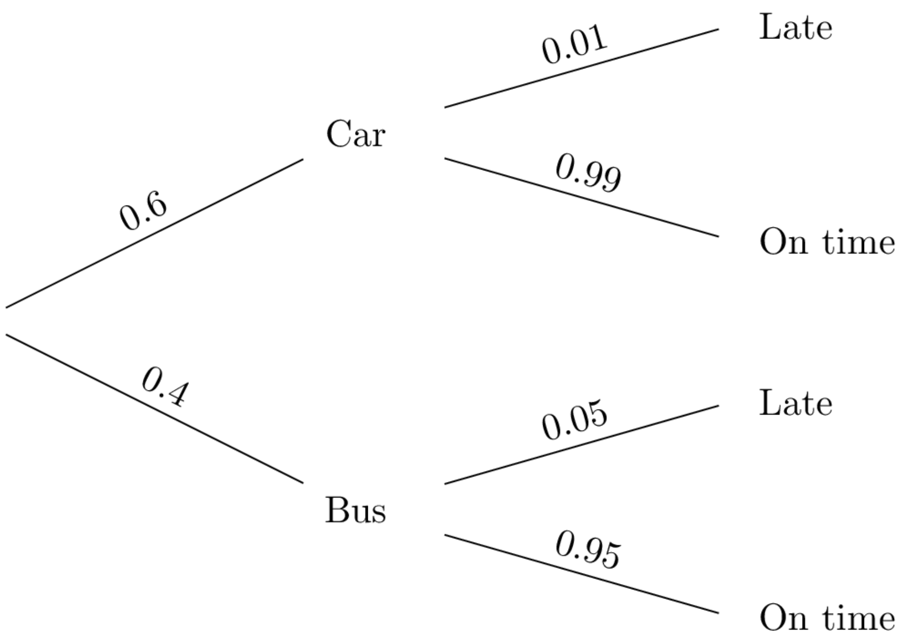
(b) Daniel took the bus, given that he was late for school.

On two randomly chosen days, find the probability that

(c) Daniel was late exactly once,

(d) Daniel took the bus on both days, given that he was late exactly once.

Solution 1.



(a) Case 1: Daniel takes a car and is on time.
Case 2: Daniel takes a bus and is on time.

$P(\text{on time}) = (0.6 \times 0.99) + (0.4 \times 0.95) = 0.974.$ ■

(b) $$\begin{aligned} P(\text{bus} \mid \text{late}) &= \frac{P(\text{bus} \cap \text{late})}{P(\text{late})} \\ &= \frac{0.4 \times 0.05}{1 - 0.974} \\ &= \frac{10}{13}. \quad \blacksquare \end{aligned}$$

(c) We find the probability that Daniel is late on day one and not late on day two (this is done by multiplication). It could also be the other way round (not late on day one and late on day two) so we apply a rearrangement of 2!.

Required probability $= 0.974 \times (1 - 0.974) \times 2! = 0.0506.$ ■

(d) After applying the conditional probability formula, we realize that we need the probability that Daniel took the bus on both days **and** ($\cap$) he was late exactly once. This is accomplished if he takes the bus and is late on one day and takes the bus and is not late on the other.

$P(\text{bus both days} \cap \text{late exactly once})$
$= (0.4 \times 0.05) \times (0.4 \times 0.95) \times 2!$
$= 0.0152.$

$P(\text{bus both days} \mid \text{late exactly once})$

$$= \frac{P(\text{bus both days} \cap \text{late exactly once})}{P(\text{late exactly once})}$$

$$= \frac{0.0152}{0.050648}$$

$= 0.300.$ ■

22.4 Venn diagram

A **Venn diagram** is a very useful tool to help us visualize sets, especially those involving complements, unions and/or intersections.

Example 2 (Venn diagram)**.**

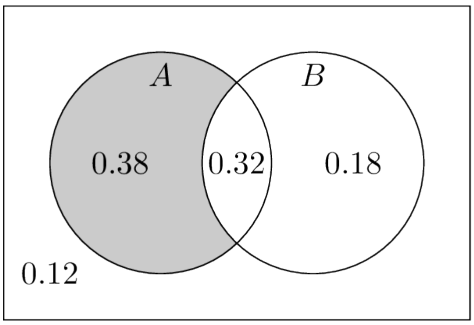
Two events A and B are such that $P(A) = 0.7, P(B) = 0.5$, and $P(A \cap B') = 0.38$.

Find the value of

(a) $P(A \cup B)$,

(b) $P(A' \mid B')$.

Solution 2.



(a) $P(A \cup B) = 0.38 + 0.5 = 0.88.$ ■

(b) $$\begin{aligned} P(A' \mid B') &= \frac{P(A' \cap B')}{P(B')} \\ &= \frac{0.12}{1-0.5} \\ &= 0.24. \quad \blacksquare \end{aligned}$$

22.5 Permutations and combinations

Probabilities can be calculated by counting the number of ways. This can be accomplished by using the techniques from the previous topic of permutations and combinations.

$$\boxed{P(A) = \frac{\text{number of ways for } A \text{ to occur}}{\text{total number of ways without restrictions}}}$$

Example 3 (permutations and combinations)**.**

Anna has 7 blouses, 3 skirts and 6 accessories, all distinct. One blouse has a floral pattern while another has a polka dot design.

(a) Anna hangs all her blouses in a line. Find the probability that the floral blouse is next to the polka dot blouse.

For Anna, an outfit consists of one blouse, one skirt and either one or two accessories and wears them.

(b) Find the total number of unique outfits that Anna has.

Her favorite accessory out of the 6 is a designer watch. When Anna is about to head out today, she picks an outfit to wear randomly such that each unique outfit has the same chance of being chosen.

(c) Find the probability that her outfit consists of either her floral blouse or her designer watch (or both).

(d) Find the probability that her outfit consists of two accessories, given that she is wearing her designer watch.

Solution 3.

(a) Using the 'grouping' method,

$$\text{probability required} = \frac{6! \times 2!}{7!} = \frac{2}{7}. \quad \blacksquare$$

(b) Case 1: exactly one accessory.
Case 2: exactly two accessories.

$$\begin{aligned}\text{Total number of ways} &= \binom{7}{1} \times \binom{3}{1} \times \binom{6}{1} + \binom{7}{1} \times \binom{3}{1} \times \binom{6}{2} \\ &= 126 + 315 \\ &= 441. \quad \blacksquare\end{aligned}$$

(c) Let A denote the event that her outfit consists of her floral blouse.
Let B denote the event that her outfit consists of her designer watch.

$P(A) = \dfrac{1}{7}$.

For $P(B)$, we consider two cases: when her designer watch is her only accessory and when her designer watch is part of two accessories.

$$P(B) = \frac{\binom{7}{1} \times \binom{3}{1} \times 1 + \binom{7}{1} \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{2}{7}.$$

$$P(A \cap B) = \frac{1 \times \binom{3}{1} \times 1 + 1 \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{2}{49}.$$

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{7} + \frac{2}{7} - \frac{2}{49} \\ &= \frac{19}{49}. \quad \blacksquare\end{aligned}$$

(d) Let C denoted the event that her outfit consists of two accessories.

$$P(B \cap C) = \frac{\binom{7}{1} \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{5}{21}.$$

$$\begin{aligned}P(C \mid B) &= \frac{P(C \cap B)}{P(B)} \\ &= \frac{\frac{5}{21}}{\frac{2}{7}} \\ &= \frac{5}{6}. \quad \blacksquare\end{aligned}$$

22.6 Table of outcomes

Table of outcomes can be very useful to visualize problem situations with two sub-events.

Example 4 (table of outcomes)**.**

Two fair six-sided dice, one blue and one red, are thrown. Let A denote the event that the blue die shows a 6. Let B denote the event that the minimum of the two dice is 3.

(a) Find the probability that either A or B occurs (but not both).

(b) Find $P(B \mid A')$.

Solution 4.

Red ↓	**Min**	**1**	**2**	**3**	**4**	**5**	**6** ← Blue
	1	1	1	1	1	1	1
	2	1	2	2	2	2	2
	3	1	2	3	3	3	3
	4	1	2	3	4	4	4
	5	1	2	3	4	5	5
	6	1	2	3	4	5	6

(a) The shaded regions on the table are the outcomes required.
Probability required $= \dfrac{11}{36}$. ■

(b) The darker shaded regions represents $B \cap A'$.

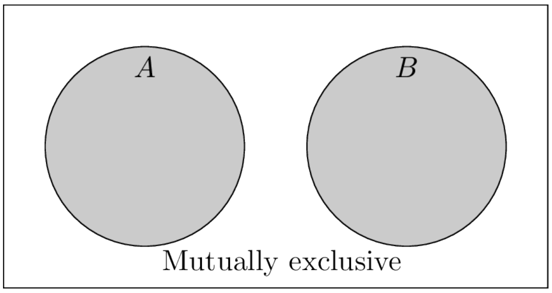
$$\begin{aligned} P(B \mid A') &= \frac{P(B \cap A')}{P(A')} \\ &= \frac{\frac{6}{36}}{1 - \frac{1}{6}} \\ &= \frac{1}{5}. \end{aligned}$$ ■

22.7 Mutually exclusive events

Two events are **mutually exclusive** if they cannot occur at the same time.

$$\boxed{P(A \cap B) = 0 \text{ for mutually exclusive events } A \text{ and } B.}$$

For example, for a single die roll, the event of rolling a 5 and the event of rolling a 3 are mutually exclusive.



22.8 Independent events

Conditional probabilities help us navigate changes in probabilities depending on the information we have. However, some information have no effect on the probability of an event. For example, if B is the event that I had a sandwich for breakfast this morning in Singapore, then knowing whether B has occurred should have no impact on the probability of event A that it rained in Australia. Thus $P(A \mid B) = P(A)$.

We can generalizing this observation and apply the conditional probability formula to arrive at the definition of **independent** events:
$\boxed{P(A \cap B) = P(A)P(B) \text{ for independent events } A \text{ and } B.}$

For example, if we roll a dice twice, the event of rolling a 5 on the first roll and the event of rolling a 3 on the second roll is typically considered to be independent.

Concept check: Note that mutually exclusivity and independence are two separate concepts. Are you able to come up with two events that are mutually exclusive but not independent? How about two events that are independent but not mutually exclusive? How about two events that are neither mutually exclusive nor independent?

Example 5 (mutually exclusive and independent events)**.**

Two fair six-sided dice, one blue and one red, are thrown. Let A denote the event that the blue die shows a 2. Let B denote the event that the sum of the two dice is 7.

(a) Explain whether the events A and B are mutually exclusive.

(b) Explain whether the events A and B are independent.

Solution 5.

Red ↓	**Sum**	**1**	**2**	**3**	**4**	**5**	**6** ← Blue
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(a) Since $P(A \cap B) = \dfrac{1}{36} \neq 0$,

A and B are not mutually exclusive. ■

(b) $P(A) = \dfrac{6}{36} = \dfrac{1}{6}$.

$P(B) = \dfrac{6}{36} = \dfrac{1}{6}$.

Since $P(A)P(B) = \dfrac{1}{36} = P(A \cap B)$,

A and B are independent. ■

22.9 Range of probabilities

In certain situations, we will have incomplete information to determine exactly all the probabilities we are interested in. In such cases, because all probabilities must be between 0 and 1, there will be range of probabilities that are possible (i.e. there will be a minimum possible and maximum possible value).

Using algebra in a Venn diagram is a useful tool to determine the range of probabilities, as shown in the following example.

Example 6 (range of probabilities)**.**

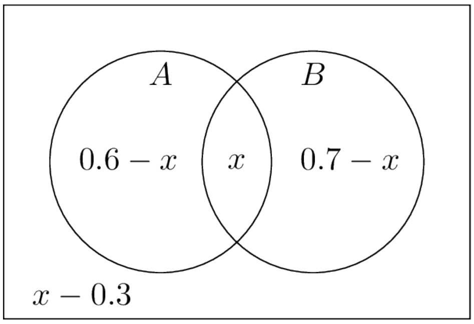
Two events A and B are such that $P(A) = 0.6$ and $P(B) = 0.7$. Find the range of possible values of $P(A \cap B)$.

Solution 6.

We let x denote $P(A \cap B)$. Then $P(A' \cap B) = 0.6 - x$ and $P(A' \cap B) = 0.7 - x$.

Since all probabilities add up to 1,
$P(A' \cap B') = 1 - (0.6 - x) - (0.7 - x) - x = x - 0.3$.

These illustrated in the Venn diagram below:



Since all probabilities must be between 0 and 1, x must be at least 0.3 otherwise $P(A' \cap B') = x - 0.3$ will be negative. Similarly, x must be at most 0.6 so that $P(A' \cap B) = 0.6 - x$ will be non-negative.

Hence $0.3 \leq P(A \cap B) \leq 0.6$. ■

22.10 Consolidated examples

Example 7 (exam style question (Venn diagram)).

For events A and B it is given that $P(A) = 0.6, P(B) = 0.9$ and $P(A \cap B) = 0.59$.

(a) Find $P(A \cup B)$.

For a third event C it is given that $P(C) = 0.7$ and that A and C are independent.

(b) Find $P(A' \cap C)$.

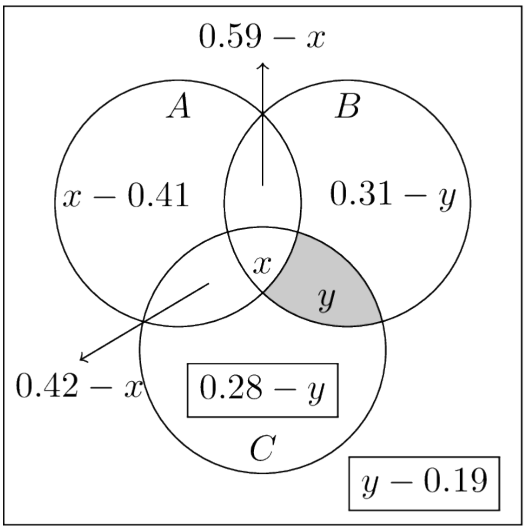
(c) Find the range of possible values of $P(A' \cap B \cap C)$.

Solution 7.

(a)
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.9 - 0.59 \\ &= 0.91. \quad \blacksquare \end{aligned}$$

(b)
$$\begin{aligned} P(A' \cap C) &= P(C) - P(A \cap C) \\ &= P(C) - P(A)P(C) \text{ since } A, C \text{ independent} \\ &= 0.7 - (0.6)(0.7) \\ &= 0.28. \quad \blacksquare \end{aligned}$$

(c) We let x denote $P(A \cap B \cap C)$ and y denote $P(A' \cap B \cap C)$ and fill in our Venn diagram:



Since all probabilities have to be between 0 and 1, by observing the probabilities that are boxed up on the Venn diagram,

$0.19 \leq P(A' \cap B \cap C) \leq 0.28.$ ∎

Example 8 (exam style question (tree diagram)).

To predict the weather, a simple computer simulation uses the following model:

The probability that it rains on the first day is 0.6. For the next two days, the probability of rain is

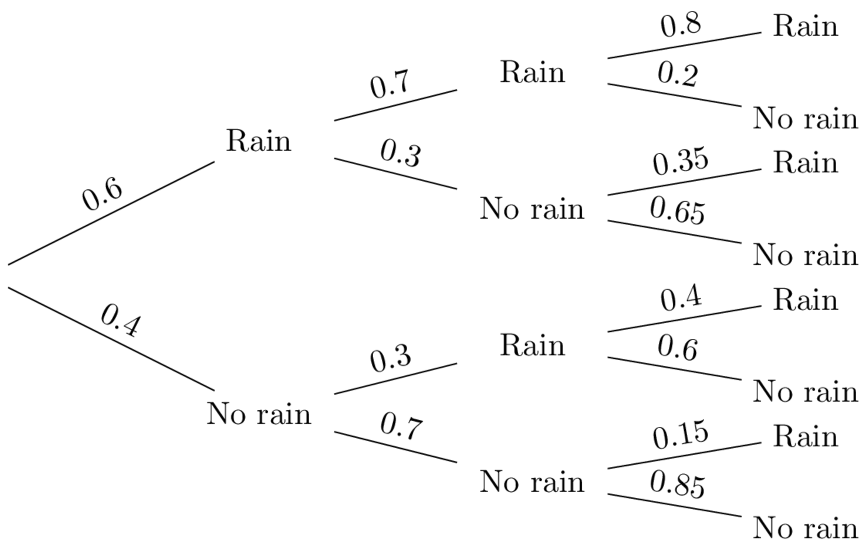
- 0.1 higher than the probability of raining the previous day if it rained the previous day,
- half the probability of raining the previous day if it did not rain the previous day.

(a) Find the probability that it rains on all three days.

(b) Find the probability that it rained on at least one day.

(c) Find the probability that it rained on the first day, given that it rained on the third day.

Solution 8.



(a) Probability required $= (0.6 \times 0.7 \times 0.8) = 0.336$. $\blacksquare$

(b) We consider the complement where it does not rain on all three days.
Probability required $= 1 - (0.4 \times 0.7 \times 0.85) = 0.762$. $\blacksquare$

(c) $P(\text{rain first} \cap \text{rain third}) = 0.336 + (0.6 \times 0.3 \times 0.35) = 0.399$.

$$\begin{aligned} P(\text{rain third}) &= 0.399 + (0.4 \times 0.3 \times 0.4) + (0.4 \times 0.7 \times 0.15) \\ &= 0.489. \end{aligned}$$

$$\begin{aligned} P(\text{rain first} \mid \text{rain third}) &= \frac{P(\text{rain first} \cap \text{rain third})}{P(\text{rain third})} \\ &= \frac{0.399}{0.489} \\ &= 0.816. \quad \blacksquare \end{aligned}$$

Links and other resources

- Online version of these notes (with less explanation text) is available at math-atlas.vercel.app/notes/probability
- Computer generated questions: math-atlas.vercel.app/questions
- Short quiz questions to test for concepts: math-atlas.vercel.app/quiz
- YouTube channel with worked TYS solutions and revision lectures tiny.cc/kelvinsoh
- Contact me at kelvinsohmath@gmail.com