22. Probability

Summary

Techniques

- $0 \le P(A) \le 1$ for all events A.
- **Complement** A': event that A does **not** occur: $\boxed{P(A') = 1 - P(A)}$.
- **Intersection**, $A \cap B$: event that A **and** B occur.
- **Union**, $A \cup B$: event that A **or** B occur (or both). $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$.
- **Subsets**: $B \subseteq A \implies P(B) \le P(A)$.
- **Conditional probability** of A **given** B, $P(A \mid B)$: $\boxed{P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}}$.
- $P(A) = \dfrac{\text{number of ways for } A \text{ to occur}}{\text{total number of ways without restrictions}}$
- The **table of outcomes**, **Venn diagram** and **tree diagrams** are useful tools in this topic.
- $P(A \cap B) = 0$ for **mutually exclusive** events A and B.
- $P(A \cap B) = P(A)P(B)$ for **independent** events A and B. Alternatively, $P(A \mid B) = P(A)$.

Example 1: table of outcomes

Example

Two fair six-sided dice, one blue and one red, are thrown. Let A denote the event that the blue die shows a 2. Let B denote the event that the sum of the two dice is 7.

(a) Explain whether the events A and B are mutually exclusive.
(b) Explain whether the events A and B are independent.

(a) Not mutually exclusive since $P(A \cap B) = \frac{1}{36} \neq 0$.

(b) Independent since $P(A \cap B) = \frac{1}{36}$ and $P(A)P(B) = \left(\frac{6}{36}\right)\left(\frac{6}{36}\right) = \frac{1}{36}$ so $P(A \cap B) = P(A)P(B)$.

Red ↓ / Sum	1	2	3	4	5	6	← Blue
1	2	3	4	5	6	7	
2	3	4	5	6	7	8	
3	4	5	6	7	8	9	
4	5	6	7	8	9	10	
5	6	7	8	9	10	11	
6	7	8	9	10	11	12	

Example 2: venn diagram, range of probabilities

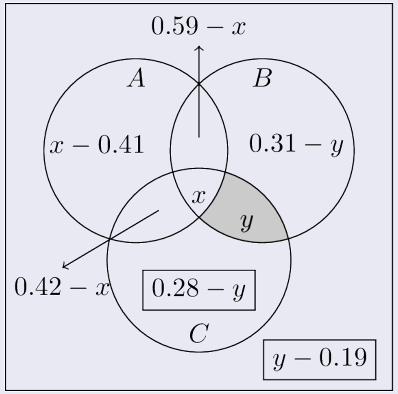
Example

Given $P(A) = 0.6, P(B) = 0.9, P(C) = 0.7, P(A \cap B) = 0.59$ and that A and C are independent, find (a) $P(A \cup B)$, (b) $P(A' \cap C)$, (c) the range of possible values of $P(A' \cap B \cap C)$.

(a) $P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$= 0.6 + 0.9 - 0.59$
$= 0.91.$

(b) $P(A' \cap C) = P(C) - P(A \cap C)$
$= P(C) - P(A)P(C)$ since A, C independent
$= 0.7 - (0.6)(0.7)$
$= 0.28.$

(c) $0.19 \le P(A' \cap B \cap C) \le 0.28.$

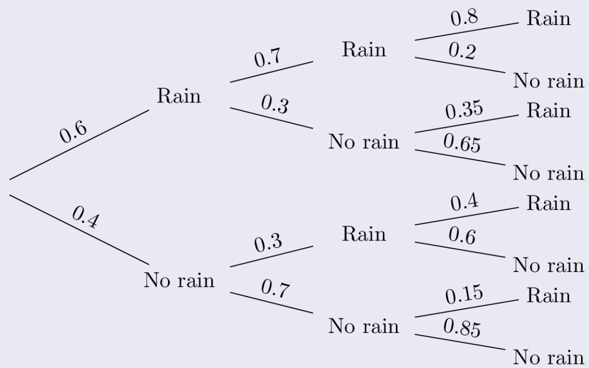


Example 3: tree diagram

Example

The probability that it rains on the first day is 0.6. For the next two days, the probability of rain is 0.1 higher than the probability of rain the previous day if it rained the previous day. If it did not rain the previous day, the probability of rain is half the probability of rain the previous day.

(a) Find the probability that it rains on all three days.
(b) Find the probability that it rained on at least one day.
(c) Find the probability that it rained on the first day, given that it rained on the third day.



(a) Probability required
$= 0.6 \times 0.7 \times 0.8 = 0.336.$

(b) Probability required
$= 1 - (0.4 \times 0.7 \times 0.85) = 0.762.$

(c) $P(\text{rain first} \mid \text{rain third})$
$= \dfrac{P(\text{rain first} \cap \text{rain third})}{P(\text{rain third})}$
$= \dfrac{0.336 + (0.6 \times 0.3 \times 0.35)}{0.489}$
$= 0.816.$