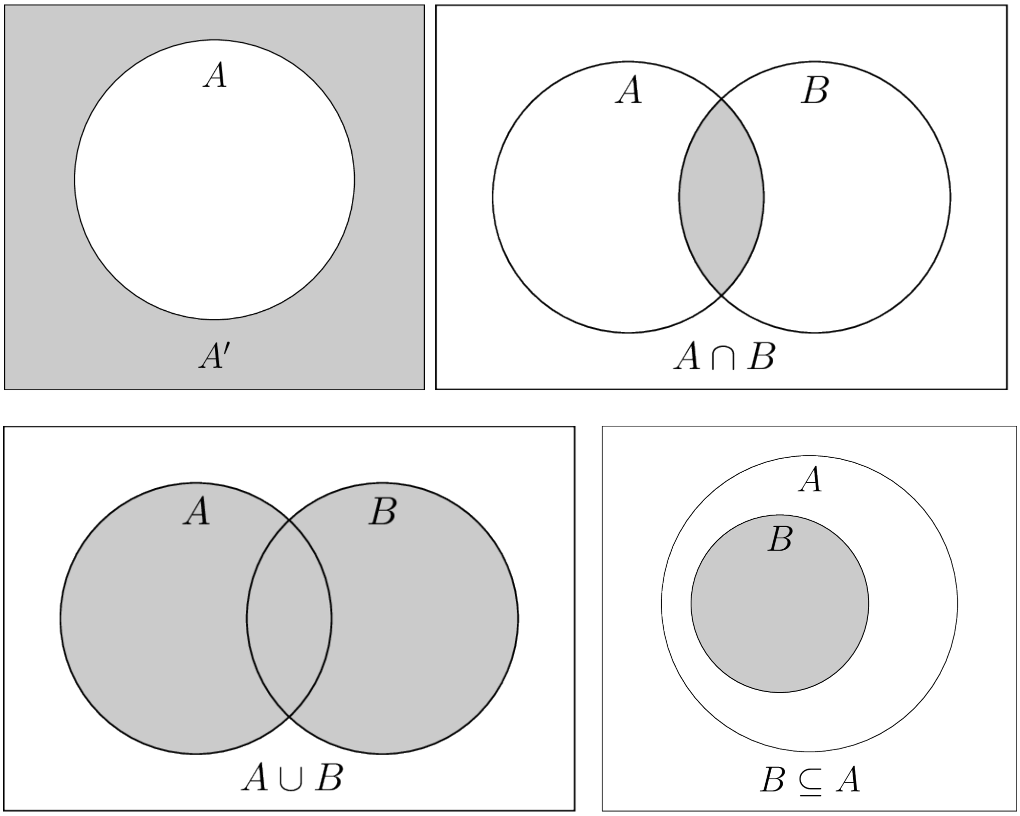
22 Probability

22.1 Basics and notation

- $0 \leq P(A) \leq 1$ for all events A.
- **Complement** A': event that A does **not** occur: $\boxed{P(A') = 1 - P(A)}$.
- **Intersection**, $A \cap B$: event that A **and** B occur.
- **Union**, $A \cup B$: event that A **or** B occur (or both).
  $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$.
- **Subsets**: $\boxed{B \subseteq A \implies P(B) \leq P(A)}$.



Adding and multiplying probabilities

Similar to the addition principle in the chapter of permutations and combinations, we **add** probabilities if we can break an event up into **cases** that **do not overlap**.

Meanwhile, similar to the multiplication principle in P&C, we **multiply** probabilities if we break an event up into **steps**.

22.2 Conditional probability

- **Conditional probability** of A **given** B, $P(A \mid B)$:
  $\boxed{P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}}$.

22.3 Tree diagram

Example 1 (tree diagram)**.**

There is a probability of 0.6 that Daniel takes his parents' car to go to school. Otherwise, he will take a bus.

If he takes his parents' car, he has a 0.01 probability of being late. The conditional probability that Daniel is late for school given that he took a bus is 0.05.

Find the probability that

(a) Daniel arrives to school on time,

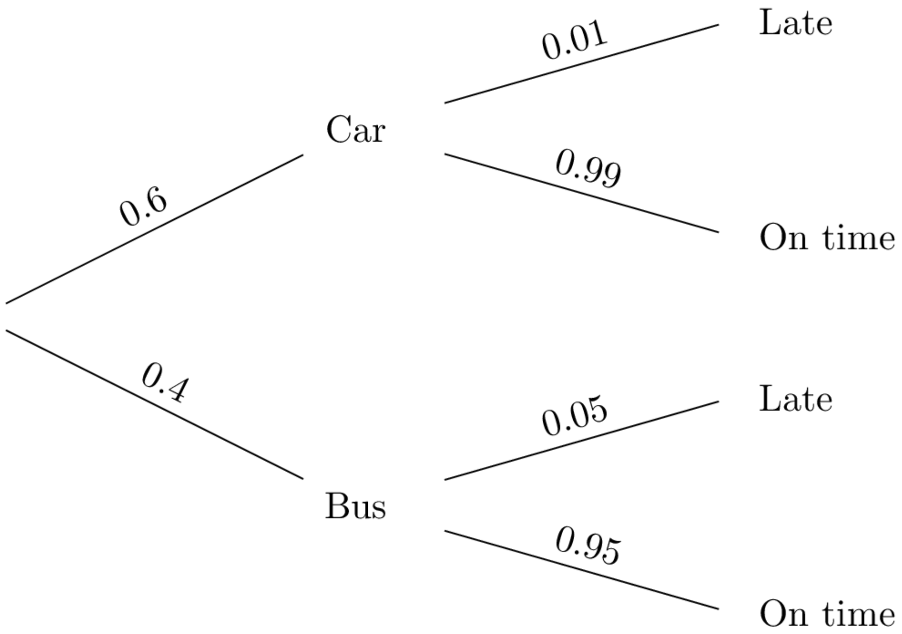
(b) Daniel took the bus, given that he was late for school.

On two randomly chosen days, find the probability that

(c) Daniel was late exactly once,

(d) Daniel took the bus on both days, given that he was late exactly once.

Solution 1.



(a) $P(\text{on time}) = (0.6 \times 0.99) + (0.4 \times 0.95) = 0.974.$ ■

(b) $$P(\text{bus} \mid \text{late}) = \frac{P(\text{bus} \cap \text{late})}{P(\text{late})} = \frac{0.4 \times 0.05}{1 - 0.974} = \frac{10}{13}.$$ ■

(c) Required probability $= 0.974 \times (1 - 0.974) \times 2! = 0.0506.$ ■

(d) $P(\text{bus both days} \cap \text{late exactly once})$
$= (0.4 \times 0.05) \times (0.4 \times 0.95) \times 2!$
$= 0.0152.$

$P(\text{bus both days} \mid \text{late exactly once})$
$$= \frac{P(\text{bus both days} \cap \text{late exactly once})}{P(\text{late exactly once})} = \frac{0.0152}{0.050648} = 0.300.$$ ■

22.4 Venn diagram

Example 2 (Venn diagram).

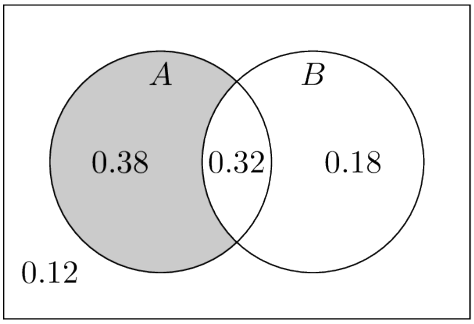
Two events A and B are such that $P(A) = 0.7, P(B) = 0.5$, and $P(A \cap B') = 0.38$.

Find the value of

(a) $P(A \cup B)$,

(b) $P(A' \mid B')$.

Solution 2.



(a) $P(A \cup B) = 0.38 + 0.5 = 0.88.$ ■

(b) $$\begin{aligned} P(A' \mid B') &= \frac{P(A' \cap B')}{P(B')} \\ &= \frac{0.12}{1 - 0.5} \\ &= 0.24. \end{aligned}$$ ■

22.5 Permutations and combinations

$$\boxed{P(A) = \frac{\text{number of ways for } A \text{ to occur}}{\text{total number of ways without restrictions}}}$$

Example 3 (permutations and combinations).

Anna has 7 blouses, 3 skirts and 6 accessories, all distinct. One blouse has a floral pattern while another has a polka dot design.

(a) Anna hangs all her blouses in a line. Find the probability that the floral blouse is next to the polka dot blouse.

For Anna, an outfit consists of one blouse, one skirt and either one or two accessories and wears them.

(b) Find the total number of unique outfits that Anna has.

Her favorite accessory out of the 6 is a designer watch. When Anna is about to head out today, she picks an outfit to wear randomly such that each unique outfit has the same chance of being chosen.

(c) Find the probability that her outfit consists of either her floral blouse or her designer watch (or both).

(d) Find the probability that her outfit consists of two accessories, given that she is wearing her designer watch.

Solution 3.

(a) Probability required $= \dfrac{6! \times 2!}{7!} = \dfrac{2}{7}$. $\blacksquare$

(b)
$$\begin{aligned}\text{Total number of ways} &= \binom{7}{1} \times \binom{3}{1} \times \binom{6}{1} + \binom{7}{1} \times \binom{3}{1} \times \binom{6}{2} \\ &= 126 + 315 \\ &= 441. \quad \blacksquare\end{aligned}$$

(c) Let A denote the event that her outfit consists of her floral blouse.
Let B denote the event that her outfit consists of her designer watch.

$$P(A) = \frac{1}{7}, \qquad P(B) = \frac{\binom{7}{1} \times \binom{3}{1} \times 1 + \binom{7}{1} \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{2}{7}.$$

$$P(A \cap B) = \frac{1 \times \binom{3}{1} \times 1 + 1 \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{2}{49}.$$

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{7} + \frac{2}{7} - \frac{2}{49} \\ &= \frac{19}{49}. \quad \blacksquare\end{aligned}$$

(d) Let C denoted the event that her outfit consists of two accessories.

$$P(B \cap C) = \frac{\binom{7}{1} \times \binom{3}{1} \times \binom{5}{1}}{441} = \frac{5}{21}.$$

$$\begin{aligned} P(C \mid B) &= \frac{P(C \cap B)}{P(B)} \\ &= \frac{\frac{5}{21}}{\frac{2}{7}} \\ &= \frac{5}{6}. \quad \blacksquare \end{aligned}$$

22.6 Table of outcomes

Example 4 (table of outcomes)**.**

Two fair six-sided dice, one blue and one red, are thrown. Let A denote the event that the blue die shows a 6. Let B denote the event that the minimum of the two dice is 3.

(a) Find the probability that either A or B occurs (but not both).

(b) Find $P(B \mid A')$.

Solution 4.

	Min	**1**	**2**	**3**	**4**	**5**	**6**	← Blue
Red	**1**	1	1	1	1	1	1	
↓	**2**	1	2	2	2	2	2	
	3	1	2	3	3	3	3	
	4	1	2	3	4	4	4	
	5	1	2	3	4	5	5	
	6	1	2	3	4	5	6	

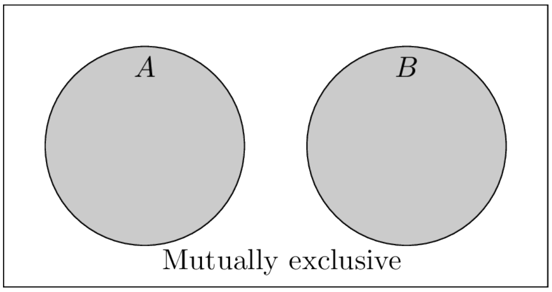
(a) Probability required $= \frac{11}{36}$. $\blacksquare$

(b) $$\begin{aligned} P(B \mid A') &= \frac{P(B \cap A')}{P(A')} \\ &= \frac{\frac{6}{36}}{1 - \frac{1}{6}} \\ &= \frac{1}{5}. \quad \blacksquare \end{aligned}$$

22.7 Mutually exclusive events

Two events are **mutually exclusive** if they cannot occur at the same time.

$$\boxed{P(A \cap B) = 0 \text{ for mutually exclusive events } A \text{ and } B.}$$



22.8 Independent events

Two events are **independent** if they satisfy the following formula:

$$\boxed{P(A \cap B) = P(A)P(B) \text{ for independent events } A \text{ and } B.}$$

We also have $P(A \mid B) = P(B)$ if A and B are independent. Intuitively, independent events describe events that do not 'affect' each other probabilistically.

Example 5 (mutually exclusive and independent events)**.**

Two fair six-sided dice, one blue and one red, are thrown. Let A denote the event that the blue die shows a 2. Let B denote the event that the sum of the two dice is 7.

(a) Explain whether the events A and B are mutually exclusive.

(b) Explain whether the events A and B are independent.

Solution 5. (a) Since $P(A \cap B) = \dfrac{1}{36} \neq 0$,

A and B are not mutually exclusive. ■

(b) $P(A) = \dfrac{6}{36} = \dfrac{1}{6}, \qquad P(B) = \dfrac{6}{36} = \dfrac{1}{6}.$

Since $P(A)P(B) = \dfrac{1}{36} = P(A \cap B)$, A and B are independent. ■

22.9 Range of probabilities

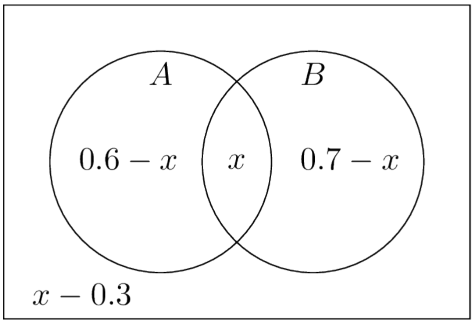
Example 6 (range of probabilities)**.**

Two events A and B are such that $P(A) = 0.6$ and $P(B) = 0.7$. Find the range of possible values of $P(A \cap B)$.

Solution 6.

We let x denote $P(A \cap B)$. Then $P(A' \cap B) = 0.6 - x$ and $P(A' \cap B) = 0.7 - x$.

Since all probabilities add up to 1,
$P(A' \cap B') = 1 - (0.6 - x) - (0.7 - x) - x = x - 0.3$.



Since all probabilities must be between 0 and 1, $0.3 \leq P(A \cap B) \leq 0.6$. ■

22.10 Consolidated examples

Example 7 (exam style question (Venn diagram))**.**

For events A and B it is given that $P(A) = 0.6, P(B) = 0.9$ and $P(A \cap B) = 0.59$.

(a) Find $P(A \cup B)$.

For a third event C it is given that $P(C) = 0.7$ and that A and C are independent.

(b) Find $P(A' \cap C)$.

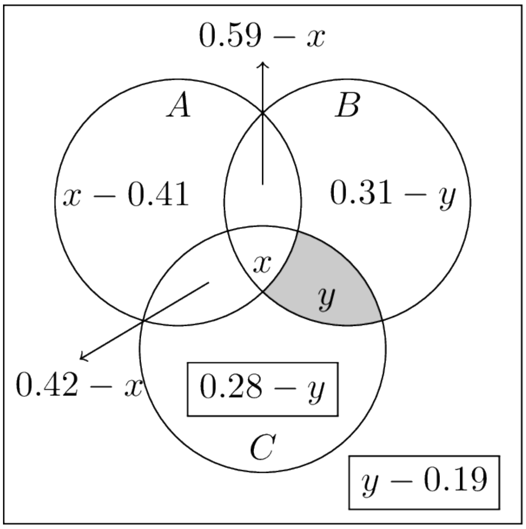
(c) Find the range of possible values of $P(A' \cap B \cap C)$.

Solution 7.

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.91.$ ■

(b) $P(A' \cap C) = P(C) - P(A \cap C)$
$= P(C) - P(A)P(C)$ since A, C independent
$= 0.28.$ ■

(c) We let x denote $P(A \cap B \cap C)$ and y denote $P(A' \cap B \cap C)$



Since all probabilities have to be between 0 and 1,
$0.19 \le P(A' \cap B \cap C) \le 0.28.$ ■

Example 8 (exam style question (tree diagram)).

To predict the weather, a simple computer simulation uses the following model:

The probability that it rains on the first day is 0.6. For the next two days, the probability of rain is

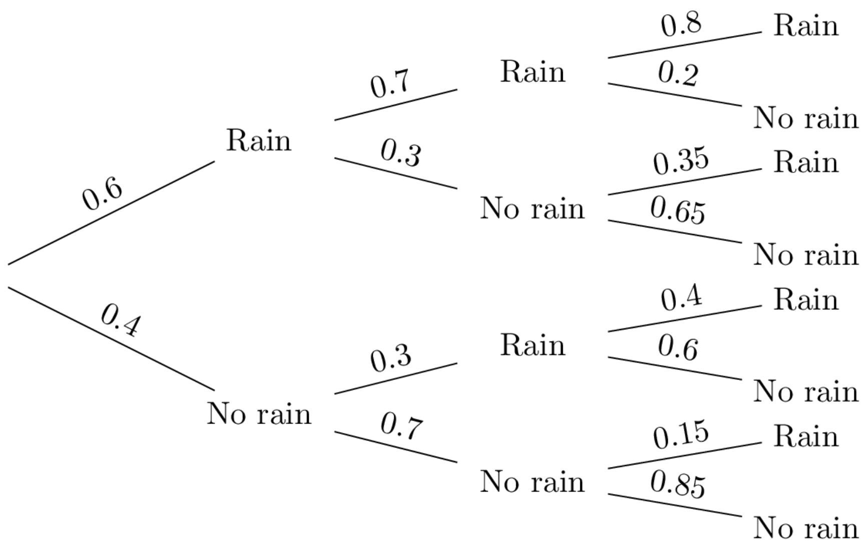
- 0.1 higher than the probability of raining the previous day if it rained the previous day,
- half the probability of raining the previous day if it did not rain the previous day.

(a) Find the probability that it rains on all three days.

(b) Find the probability that it rained on at least one day.

(c) Find the probability that it rained on the first day, given that it rained on the third day.

Solution 8.



(a) Probability required $= (0.6 \times 0.7 \times 0.8) = 0.336$. ■

(b) We consider the complement where it does not rain on all three days.
Probability required $= 1 - (0.4 \times 0.7 \times 0.85) = 0.762$. ■

(c) $P(\text{rain first} \cap \text{rain third}) = 0.336 + (0.6 \times 0.3 \times 0.35) = 0.399$.

$$\begin{aligned} P(\text{rain third}) &= 0.399 + (0.4 \times 0.3 \times 0.4) + (0.4 \times 0.7 \times 0.15) \\ &= 0.489. \end{aligned}$$

$$\begin{aligned} P(\text{rain first} \mid \text{rain third}) &= \frac{P(\text{rain first} \cap \text{rain third})}{P(\text{rain third})} \\ &= \frac{0.399}{0.489} \\ &= 0.816. \quad \blacksquare \end{aligned}$$